BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

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VOLUME XXXVII, NUMBER 8

AUGUST, 1931

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1931

PUBLISHED MONTHLY

Whole No. 383 $9.00 a Year

Entered as second class matter April 24, 1926 at the post office at Menasha, Wis., under the act of August 24, 1912.

PUBLICATIONS OF THIS SOCIETY

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SOME RECENT DEVELOPMENTS IN ABSTRACT ALGEBRA*

BY OYSTEIN ORE

1. *Introduction.* If one should try to define algebra, it might be said that algebra deals with the formal combination of symbols according to prescribed rules. Such formal combinations are, however, obviously fundamental in most branches of mathematics even outside algebra in the ordinary sense. The recognition of this formal element in the mathematical theories has naturally led to an *algebraization*, which can easily be observed in the present state of many domains of mathematics; if one adopts the views of Hilbert, the whole system of mathematics can be formalized in this way.

When a certain number of formal operations have been laid down, a principal problem is to determine the structure of systems which are *closed* with respect to these operations, that is, have the property that any operation on elements again gives an element of the system. In this problem the notion of *isomorphism* is fundamental; two systems S and S' both closed with respect to a given system of operations, are said to be isomorphic or abstractly identical with respect to these operations if there exists a one-to-one correspondence between the elements of S and S', such that any formal combination of elements in S corresponds to the analogous construction with corresponding elements in S'. An isomorphism between elements of the same system is called an *automorphism*. Two isomorphic systems are in algebra considered as equivalent, and it could therefore also be said, that algebra deals with those properties of systems, which are invariant for isomorphisms. Abstract systems and the notion of isomorphisms originated in the theory of finite groups, where the properties of groups were studied independently of the particular representation of the group.

In an algebraic system *equality* is usually defined and satisfies the following axioms:

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* An address presented to the Society at the request of the program committee, December 31, 1930.

E. Axioms of Equality

E I. *Determination.* For two arbitrary elements, either $a=b$ or $a\neq b$.

E II. *Reflexitivity.* $a=a$.

E III. *Symmetry.* From $a=b$ follows $b=a$.

E IV. *Transitivity.* From $a=b$, $b=c$ follows $a=c$.

Every such definition of equality, which is ordinarily not unique for the given system, constitutes a division of the elements into classes.

Usually algebra deals with systems which are closed with respect to one or two operations, addition and multiplication, satisfying all or some of the following axioms:

A. Axioms of Addition

A I. For two arbitrary elements a and b in S there exists a sum $a+b$, which is a uniquely defined element of S.

A II. *Equality.* If $a=b$ and $a_1=b_1$, then $a+a_1=b+b_1$.

A III. *Associative law.* $a+(b+c)=(a+b)+c$.

A IV. *Zero-element.* There exists an element 0 for which $0+a=a+0=a$ for all a.

A V. *Subtraction.* To every element a there exists another $-a$ such that $a+(-a)=0$.

A VI. *Commutative law.* $a+b=b+a$.

M. Axioms of Multiplication

M I. For two arbitrary elements a and b there exists a product $a\cdot b$ which is a uniquely defined element of S.

M II. *Equality.* From $a=b$, $a_1=b_1$, follows $aa_1=bb_1$.

M III. *Associative law.* $a(bc)=(ab)c$.

M IV. *Distributive law.* $(b+c)a=ba+ca$, $a(b+c)=ab+ac$.

M V. *Converse equality axiom.* From $ab=ac$ or $ba=ca$ follows $b=c$, when $a\neq 0$.

M VI. *Unit element.* There exists an element $\mathcal{E}$ for which $\mathcal{E}a=a\mathcal{E}=a$ for all a.

M VII. *Division.* For every $a\neq 0$ there exists an a^{-1} such that $a\cdot a^{-1}=\mathcal{E}$.

M VIII. *Commutative law.* $ab=ba$.

By various choices among these axioms one obtains a series of different types of algebraic systems. Among the main types I

shall only mention the *moduli* (satisfying ordinarily A I–A VI), *groups* (M I–M VII except M IV), *rings* or *algebras* (satisfying A I–A VI and at least M I, M II) and finally *fields* or *division-algebras* satisfying all axioms except possibly M VIII. A survey of the nomenclature can be found in a recent paper by E. T. Bell.*

In the following I shall have to limit my topic mainly to the *commutative fields*,† that is, systems which satisfy *all* axioms mentioned above. Among the various algebraic sets these form a particularly important group, and they are also one of the few systems for which the principal features of the structure have been determined.

The theory of fields is closely connected with some of the most important problems of algebra.

Examples of fields are abundant: The sets of all rational, algebraic, real or complex numbers satisfy the axioms of a field; so do the sets of all rational or algebraic functions of one or more variables with coefficients in a given field. An example of a field with only a finite number of elements is the system of all remainders (mod p), where p is an ordinary prime. I observe here, that the system of axioms for a field enumerated above is not *reduced*, that is, some axioms are consequences of others. Various reduced systems have been proposed by Dickson‡ and Huntington.§

Closely connected with the fields are the *domains of integrity*, which are rings satisfying all multiplication-axioms except M VI and M VII. An important property of these rings is the following: If a and $b \neq 0$ are elements of a domain of integrity D, one can construct the formal quotients (a/b) and define addition and multiplication for them much in the same way as the rational numbers are derived from the integers. The set of all these fractions then is a field K containing D, and K is usually called the *quotient-field* of D.

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* E. T. Bell, *Unique decomposition*, American Mathematical Monthly, vol. 37 (1930), pp. 400–418.

† In the following the word "field" always stands for "commutative field."

‡ L. E. Dickson, Transactions of this Society, vol. 4 (1903), pp. 13–20; ibid., vol. 6 (1905), pp. 198–204.

§ E. V. Huntington, Transactions of this Society, vol. 4 (1903), pp. 31–37; ibid., vol. 6 (1905), pp. 181–197.

2. *Characteristics of a Field.* The necessity for a classification of all fields was first indicated by Weber* in the discussion of the following fundamental problem. Let K be a field and let us consider equations with coefficients in K. For which fields K does the Galois theory of equations hold? I shall only mention here, without stating the problem in a precise form at present, that not for all fields and equations is it possible to develop the Galois theory in its ordinary form.

The solution of this problem is mainly due to Steinitz† (using also ideas of Dedekind)‡ and is contained among a series of other results in his classical paper on the structure of fields.

All fields can, according to Steinitz, be divided into two principal types. Thus, let K be an arbitrary field and let us consider all subfields of K; the elements which these fields have in common also form a field, the prime-field P of K, which is contained in all other subfields of K. It is easy to see, that only two types of prime-fields are possible; the unit element obviously is contained in P and P therefore also contains all elements

$$\mathcal{E}, 2\mathcal{E}, \cdots, n\mathcal{E}, \cdots.$$

These elements are either all different or there exists a rational integer p such that $p\mathcal{E}=0$. In the first case P must also contain all fractions $(n\mathcal{E}/m\mathcal{E})$ and since these elements form a field one sees that $P=P_0$ is isomorphic to the field of rational numbers. K is then said to have the *characteristic* 0.

When on the other hand $p\mathcal{E}=0$, it follows easily that the smallest p must be a prime, and $P=P_p$ is in this case isomorphic to the finite field mentioned above consisting of the p residues (mod p). We then say that K has the *characteristic* p. Any subfield of K has the same characteristic as K, and when the characteristic is a prime p, we have $pa=p\mathcal{E}a=0$, that is, all expressions in K can be reduced (mod p).

3. *Algebraic and Transcendental Adjunctions.* From the

* H. Weber, *Die allgemeine Grundlagen der Galoisschen Gleichungstheorie*, Mathematische Annalen, vol. 43 (1893), pp. 521–549.

† E. Steinitz, *Algebraische Theorie der Körper*, Journal für Mathematik, vol. 137 (1910), pp. 167–309. This paper has recently been edited in book form and annotated by H. Hasse and R. Baer, 1930.

‡ Compare, for example, supplements to Dirichlet-Dedekind's *Zahlentheorie*.

prime-field the original field K can be obtained by the process of *adjunction*. If a is an element in K not contained in P, then all rational functions $R(a)$ of a with coefficients from P form a field $P(a)$ contained in K and containing P. If $P(a)$ is a proper subfield of K, we adjoin another element b not contained in $P(a)$ and obtain the field $P(a, b)$ etc. In this way K can be built up, if necessary by using a well-ordering of K, that is, assuming the axiom of Zermelo.

Let us therefore in general consider a field K and the possible adjunctions to K. We shall first consider the *simple* adjunctions, that is, fields obtained by the adjunction of a single element. As we shall see, there exist two principal types of simple adjunctions.

Let x be the element adjoined; then the enlarged field must contain all expressions of the form

$$a_0 + a_1 x + \cdots + a_n x^n, \tag{1}$$

where the a_i are elements of K. These expressions might be all different or some of them might be equal. In the first case an expression (1) can only be equal to zero if all coefficients vanish; these polynomials then obviously form a domain of integrity, and the quotient-field $K(x)$, consisting of all rational functions of x with coefficients in K, is the least field containing K and x. Such an adjunction is called a *transcendental adjunction*, and $K(x)$ is called a *transcendental enlargement* of K. The following fact is then obvious:

To every field K at least one simple transcendental enlargement exists, and all such fields are equivalent, that is, there exists an isomorphism between them having the particular property that the elements of K correspond to themselves.

For, if $K(y)$ is another transcendental enlargement, we can let $R(x)$ correspond to $R(y)$ for all rational functions R.

In the second case, where some of the expressions (1) are equal, there must exist a relation of the following form (writing α for x to distinguish from the former case)

$$f(\alpha) = \alpha^m + b_1\alpha^{m-1} + \cdots + b_m = 0. \tag{2}$$

If $f(\alpha)$ is the polynomial of lowest degree having this property, $f(x)$ must be an irreducible polynomial in K. Every expression (1) is equal to a reduced expression

$$c_1\alpha^{m-1} + c_2\alpha^{m-2} + \cdots + c_m, \tag{3}$$

and two expressions $A(\alpha)$ and $B(\alpha)$ are only then equal when

$$A(\alpha) \equiv B(\alpha), \qquad (\text{mod } f(\alpha)).$$

It is easily seen that all the expressions (3) form a field; to prove, for example, that corresponding to every $A(\alpha) \neq 0$ there exists an $A(\alpha)^{-1}$, we determine $G(x)$ and $H(x)$ by Euclid's algorithm such that

$$G(x)A(x) + H(x)f(x) = 1.$$

Since $f(x)$ is irreducible, this can always be accomplished, and we obtain $G(\alpha) = A(\alpha)^{-1}$ since

$$G(\alpha)A(\alpha) \equiv 1, \qquad (\text{mod } f(\alpha)).$$

The field $K(\alpha)$ is called a *simple algebraic enlargement* of K. In $K(\alpha)$ the irreducible polynomial $f(x)$ in K has a root, and conversely, when an arbitrary irreducible $f(x)$ in K is chosen, this procedure gives a method for constructing a $K(\alpha)$ in which $f(x) = 0$ has a solution.

If $f(x)$ is an irreducible polynomial in a field K there always exists a simple algebraic enlargement $K(\alpha)$ such that $f(x) = 0$ has a root in $K(\alpha)$, and all other fields having this property must have a subfield equivalent to $K(\alpha)$.

4. *Degree of Transcendency. Algebraically Complete Fields.* The last simple theorem replaces in modern algebra for most applications the following so-called *fundamental theorem of algebra*: In the field of all complex numbers every polynomial has a root.

From the results of §3 it follows that every field can be obtained from its prime-field by a series of algebraic and transcendental adjunctions. One of the fundamental results of Steinitz is this: *Every field can be obtained by first making a series of transcendental adjunctions and then a series of algebraic adjunctions to this purely transcendental field.* The number t of transcendental adjunctions required is called the *degree of transcendency* of K and is, like the characteristic, a characterizing invariant of the field. The degree of transcendency t may be infinite and in this case the cardinal number of the set of transcendental elements to be adjoined is the invariant.

Since the properties of the purely transcendental fields are fairly simple, we shall take more interest in the algebraic adjunctions. A field K' is said to be an *algebraic* enlargement of K, if it can be obtained from K by successive simple algebraic adjunctions. This is equivalent to saying that all elements in K' shall satisfy irreducible equations in K. The algebraic enlargements are again of two kinds: (a) the *finite* algebraic fields over K, which require only a finite number of adjunctions, and (b) the *infinite*, only obtainable by an *infinite* number of simple algebraic adjunctions to K. The finite algebraic fields over K are also equivalent to the fields of *finite rank* over K, that is, there exists a basis

$$\omega_1, \omega_2, \cdots, \omega_n$$

of the field, such that every number α in the field can be expressed as

$$\alpha = a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n,$$

where the a_i are elements of K.

We saw in §3 that every field K had transcendental enlargements. It is interesting to notice that not every field can be enlarged by algebraic adjunctions. A field of this kind is, for instance, the field of all complex numbers, in which no irreducible polynomials of higher degree than the first can exist. The field of all algebraic numbers has the same property. A field in which every polynomial (with coefficients in the field) reduces to a product of linear factors is called *algebraically complete*. These are obviously the only fields to which no algebraic adjunctions can be made. Steinitz was able to prove the following theorem.

For every field K there exists a unique algebraically complete field K', which is algebraic over K. All other algebraically complete fields over K contain a subfield equivalent to K'.

5. *Galois Fields and Group Theory for Abstract Fields.* We shall now return to the general problem of determining the fields to which the Galois theory can be extended. A *Galois field* $\overline{K}$ over a given field K is defined as an algebraic field in which every irreducible polynomial $f(x)$ in K either remains irreducible or decomposes into a product of linear factors. A *simple* Galois field is the field of least rank in which a given irreducible polynomial $f(x)$ decomposes into linear factors.

To every Galois field $\overline{K}$ there exists a set of automorphisms, such that when α corresponds to another element α' (the conjugates of α in the ordinary Galois theory) also contained in $\overline{K}$, then $\alpha \pm \beta$, $\alpha\beta$ will correspond to $\alpha' \pm \beta'$, $\alpha'\beta'$, while all elements of K shall remain unchanged. These automorphisms form a group G, which is the *Galois group* of $\overline{K}$ with respect to K. The well known main theorem of the Galois theory is then the following.

There exists a unique correspondence between the subgroups of G and the subfields of $\overline{K}$ (containing K), such that to any subgroup G' corresponds a subfield K' consisting of all elements left invariant by G', and conversely.

Even for a *simple* Galois field, this theorem does not always hold. It can be shown that a necessary and sufficient condition is that $f(x)$ shall have no equal roots. Steinitz says, that an irreducible polynomial $f(x)$ is of the *first kind*, when all its roots are different, and of the *second kind*, when some of them are equal. It can then be shown, that *the main theorem will only hold for Galois fields, which can be obtained from K by a series of adjunctions of the first kind.*

At first glance it seems surprising that an irreducible polynomial can have equal roots, since by the ordinary procedure one would then find that $f(x)$ had a factor in common with $f'(x)$. For certain fields of characteristic p it can happen, however, that $f'(x)$ vanishes identically. To take a simple example, let us consider the field $K = P_p(t)$ consisting of all rational functions of t with rational coefficients (mod p). In this field

$$f(x) = x^p - t$$

is irreducible and nevertheless we have

$$f(x) = (x - t^{1/p})^p, \quad f'(x) = 0.$$

Fields in which all polynomials are of the first kind have been called *separable.** All fields of characteristic 0 are separable, and among the fields of characteristic p only those are separable in which the pth root of every element again belongs to the field. The prime-field P_p is separable since by Fermat's theorem $a^p = a$.

* Steinitz says "vollkommen." I prefer the term separable introduced by B. L. van der Waerden.

These problems are closely connected with the so-called theorems of the *primitive element* (Abel's theorem) and the *finite number of subfields*. In the fields of ordinary algebraic numbers we have the theorem, that if a field $K'=K(\alpha, \beta, \cdots, \delta)$ is obtained from an algebraic field K by the adjunction of a finite number of algebraic numbers, then there exists in K' a primitive algebraic number such that $K'=K(\theta)$, and K' will only have a finite number of subfields containing K. The same theorems can be shown to hold for a general field K for a finite number of adjunctions of the first kind, and under certain conditions, specified by Steinitz, for adjunctions of the second kind. That these theorems do not hold for arbitrary algebraic adjunctions can be seen by the following example. Let $K=P_p(x, y)$ be the field obtained from the prime-field P_p by two transcendental adjunctions. Then the field $K'=K(x^{1/p}, y^{1/p})$ has no primitive element. Any element $R=R(x^{1/p}, y^{1/p})$ will always satisfy an equation

$$R^p = \overline{R}(x, y),$$

where $\overline{R}(x, y)$ is an element of K. If, therefore, it were true that

$$x^{1/p} = f_1(R), \quad y^{1/p} = f_2(R),$$

we would obtain

$$x = \bar{f}_1(\overline{R}), \quad y = \bar{f}_2(\overline{R}),$$

and there would exist an algebraic relation between x and y, contrary to hypothesis. The existence of an infinite number of subfields between K and K' can also easily be shown.

For fields in which the Galois theory holds, most of the ordinary conclusions can be drawn in an analogous way. In certain cases the fields of characteristic p present difficulties; for example, the general quadratic equation

$$x^2 + u_1 x + u_2 = 0$$

cannot be solved by radicals in a field of characteristic 2.

6. *Infinite Galois Fields.* In §5 the conditions for the validity of the Galois theory have been completely determined for all Galois fields of finite rank. Let us now consider the Galois theory of Galois fields of infinite rank. The first investigations

of this kind are due to Dedekind* who in a well known paper considered the permutations of the field of all algebraic numbers. This field has a set of automorphisms of the power of the continuum, and by a special example Dedekind was able to prove that not all subgroups will correspond to fields.

Dedekind also considered, without great success however, the automorphisms of the field of all complex numbers, and made the conjecture that the only automorphism of this field besides the identity was the correspondence between conjugate elements. From some later investigations by Ostrowski† it follows that this is not correct, since any automorphism of the field of all algebraic numbers can be extended to an automorphism of the field of complex numbers.

The general theory of groups of infinite Galois fields (of the first kind) was developed with remarkable completeness by Krull.‡ He was able to show that even for Galois fields obtained by a countable number of adjunctions, the group has the power of the continuum, and while every subfield corresponds to a subgroup, not all subgroups will correspond uniquely to a subfield.

In order to find out which subgroups would correspond to subfields, Krull introduced various topological notions for groups, like limits for a set of automorphisms, limit points and neighborhoods. The main result can then be stated briefly as follows.

A necessary and sufficient condition that a subgroup corresponds to a field in the way indicated by the main theorem is that the subgroup be closed in the topological sense, that is, contain all its limit-substitutions.

It will lead too far to go into further details about these interesting investigations; I shall only mention that for non-commutative fields also groups of automorphisms can be introduced and various remarkable results can be deduced.

7. *The Notion of Absolute Value.* Among the various other

* R. Dedekind, *Ueber die Permutationen des Körpers aller algebraischen Zahlen*, Festschrift der Gesellschaft der Wissenschaften, Göttingen, 1901. Werke, vol. 2.

† A. Ostrowski, *Ueber einige Fragen der allgemeinen Körpertheorie*, Journal für Mathematik, vol. 143 (1913), pp. 255–284.

‡ W. Krull, *Galoissche Theorie der unendlichen algebraischen Erweiterungen*, Mathematische Annalen, vol. 100 (1928), pp. 687–698.

investigations concerning fields, I shall only mention the analysis of the notion of *absolute value*.* Kürschak† defines the absolute value $\|\alpha\|$ of a number α in a given field K as a real number, such that $\|0\| = 0$, $\|\alpha\| > 0$ when $\alpha \neq 0$. Furthermore, we have

$$\|\alpha\beta\| = \|\alpha\| \cdot \|\beta\|$$

and

$$\|\alpha + \beta\| \leqq \|\alpha\| + \|\beta\|. \tag{4}$$

The absolute value corresponds in many ways to the *metric* in the theory of point sets; the inequality (4) corresponds to the triangular inequality.

When the absolute value is defined, convergence and limits can be introduced in the ordinary way. A series

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots \tag{5}$$

is said to be convergent if it satisfies $\|a_n - a_{n+k}\| < \epsilon$ for arbitrary ϵ and sufficiently large n; the series (5) has a limit a if $\|a_n - a\| < \epsilon$.

Kürschak calls a field *perfect* if all convergent series have limits in the field. A perfect field can always be obtained by adjoining the convergent series to the field, corresponding to the construction of the real field from the rational field. Kürschak's main theorem is as follows.

Every field in which an absolute value is defined, can be enlarged to a perfect, algebraically closed field.

An example is the rational field and the complex field.

Ostrowski‡ has in a number of papers continued the investigations of Kürschak; one of his results is that the least algebraically closed field K' of a perfect field K can only be perfect if K' is finite with respect to K.

The possible absolute values which can be defined for a given field are by no means unique. Let us consider the rational field R.

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* Corresponding to the German "Bewertung."

† I. Kürschak, *Ueber Limesbildung und allgemeine Körpertheorie*, Journal für Mathematik, vol. 142 (1913), pp. 211–253.

‡ A. Ostrowski, Journal für Mathematik, vol. 143 (1913) pp. 255–284; ibid., vol. 147 (1917), pp. 191–204; Acta Mathematica, vol. 41 (1918), pp. 271–284.

It can be shown that there exist two and only two principal types of absolute values. One is obtained by putting

$$\|r\| = |r|^{\rho},$$

where $0 \leqq \rho \leqq 1$ is a fixed constant and $|r|$ the ordinary absolute value. The corresponding perfect field is the field of all real numbers. The second type is obtained in the following way. Let p be a rational prime; every rational number r can then be written in the form

$$r = p^{\alpha}\frac{u}{v}, \qquad (u, v) = 1,$$

where u and v are not divisible by p and α is a positive or negative integer. Then the definition

$$\|r\| = c^{\alpha}, \qquad (0 \leqq c \leqq 1),$$

satisfies all the conditions of an absolute value. The corresponding perfect field is Hensel's field of p-adic numbers; historically it was the properties of this field which gave the impulse to a general study of abstract values. It can be shown that for arbitrary fields also there exist only the same types of absolute values as for the rational fields.*

Another interesting application of the ideas of abstract algebra is the characterization of formal-real fields given by Artin† and Schreier. A field K is said to be formally real if one can conclude from

$$\alpha^2 + \beta^2 + \cdots + \lambda^2 = 0, \qquad (\alpha, \beta, \cdots, \lambda \text{ in } K),$$

that $\alpha = \beta = \cdots = \lambda = 0$. Among the many consequences of this theory are the solution of Hilbert's problem about the possibility of representing positive-definite functions as the sum of squares and the results of Landau on the representation of numbers as the sum of squares in total-real fields.

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* Some further applications of this theory can be found by K. Rychlik, Journal für Mathematik, vol. 153 (1924), pp. 94–107.

† E. Artin and O. Schreier, *Algebraische Konstruktion reeller Körper*, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 5 (1927), pp. 85–99.

PROOF OF A WARING THEOREM ON FIFTH POWERS*

BY L. E. DICKSON

In 1770, E. Waring conjectured that "every number is a sum of nine cubes, also is a sum of 19 fourth powers, and so on to infinity."

By elaborate and very delicate analysis, Hardy and Littlewood obtained remarkable asymptotic results. Their proofs apply only to numbers exceeding an extremely large limit L.

I shall here explain my method to cover the numbers $<L$ and hence finally obtain universal theorems holding from 1 to infinity.

For greater clearness, I shall first consider briefly the case of cubes. Jacobi inspired the construction of tables showing the least number of positive cubes (or fourth powers) whose sum yields the tabulated number. In 1851, he published the table to 12,000 by Z. Dahse. In 1903, von Sterneck made a table to 40,000 and found not merely that nine cubes always suffice, but also the more vital fact that all numbers between 8,042 and 40,000 are sums of six cubes.

I shall prove that every number N lying between 40,000 and 67,000 is a sum of seven cubes. From all these numbers we subtract 27,000, which is the cube of 30. Take 27,000 from 40,000 and you have 13,000. Take 27,000 from 67,000 and you have 40,000. Since $N-30^3$ therefore lies between 13,000 and 40,000, it is a sum S of six cubes by the result quoted. Hence $N=30^3+S$ is a sum of seven cubes.

Similarly, every number M lying between 67,000 and 94,872 is a sum of seven cubes, since $M-38^3=M-54{,}872$ lies between 12,128 and 40,000 and hence is a sum of six cubes, so that M is a sum of seven cubes.

This argument may be repeated until we have finally subtracted the cube of 103. Our result is that all numbers between 8,042 and 1,132,727 are sums of seven cubes.

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* An address, presented to the Society at the request of the program committee, April 4, 1931. The presentation was made with the aid of lantern slides.

Evidently a like argument shows that still larger numbers are sums of eight cubes, in fact, all between 454 and 232,604,691.

A final such argument shows that all numbers up to 4 billion are sums of nine cubes.

We obtained this conclusion quite simply from facts in the table to 40,000. We next indicate its importance for the general theory of cubes. In 1909, Wieferich proved that all numbers exceeding $2\frac{1}{4}$ billions are sums of nine cubes. Since our two limits overlap, we finally conclude that all numbers are sums of nine cubes. We depend upon the advanced mathematical proof for very large numbers and depend upon our deductions from a table for the smaller numbers.

I now pass to my recent analogous investigations on sums of fifth powers. I made a table which shows the least number (called minimum) of fifth powers whose sum is the tabulated number less than 67,232.

We shall make use only of the part of the table after 32,107. In this part there occurs no minimum 19 or greater. After 50,425, there occur only the following ten numbers N having the minimum 18:

54430, 54738, 54770, 55211, 55276, 55393, 55571, 56057, 63769, 63801.

To each N we add $7^5=16{,}807$ and get the only possible minima $\geqq 19$ between 67,232 and 84,039. To show that each such sum $N+7^5$ actually has a minimum $\leqq 18$, we note that $N+7^5-8^5=N-15961$ has the values 38469, $\cdots$, 47840, which are seen by our table to have minima $\leqq 17$. Transposing 8^5, we conclude that $N+7^5$ has a minimum $\leqq 18$. This proves that not merely in our table, but also in an imaginary extension of it to 84,039, there occurs no minimum $\geqq 19$.

If we repeat this argument, adding 8^5 instead of 7^5 and subtracting 9^5 instead of 8^5, we secure the further extension of our result to 100,000. Hence finally we conclude that all numbers from 32,017 to 100,000 are sums of 18 fifth powers.

Just as we ascended from 6 to 7 cubes, so here we ascend from 18 to 19 fifth powers. To prove that every number N between 100,000 and $100{,}000+9^5=159{,}049$ is a sum of 19 fifth powers, note that $N-9^5$ lies between 40,951 and 100,000 and hence by our preceding result is a sum of 18 fifth powers. Similarly, we may add 10^5 and 11^5 instead of 9^5, and obtain the result that all

numbers from 32,017 to $100,000+11^5=261,051$ are sums of 19 fifth powers.

We may obtain this final limit at once, without employing our three steps, and similarly avoid the many steps used for cubes. To do this we apply the following theorem with $l=32,107$, $g=100,000$, $k=19$, $m=10$.

THEOREM. *Grant that every number n between l and g is a sum of $k-1$ fifth powers. Find the maximum integer m such that $(m+1)^5-m^5<g-l$. Then every number between l and $g+(m+1)^5$ is a sum of k fifth powers.*

To ascend to 20 fifth powers, apply the theorem with $l=32,017$, $g=261,051$, $k=20$. Now $m=14$, whence every number between l and $g+15^5=1,020,426$ is a sum of 20 fifth powers. Similarly, all numbers between 32,017 and

5,104,527	are	sums	of 21
38,658,959	"	"	" 22
456,854,452	"	"	" 23
9,496,062,420	"	"	" 24.

Continuations of these results are best made by an older theorem which is convenient for logarithmic work. It would have been very wasteful for the preceding ascents, but gives nearly as good results for the future ascents. This older theorem is the following: If numbers $<L$ are sums of n fifth powers, we compute the limit under which numbers are sums of $n+1$ fifth power by multiplying $L/5$ by its fourth root. Evidently these limits will increase faster and faster as we ascend and yield limits too large to write out in full, so that we shall merely tell the number of figures involved in them. The most significant results are the following two: All numbers having fewer than 122 figures are sums of 37 fifth powers. All having fewer than 454 figures are sums of 43 fifth powers.

Hardy and Littlewood, after years of work, finally succeeded in proving, by very long and most delicate analysis, that all numbers involving more than 445 figures* are sums of 54 fifth powers. Combining this result with our preceding one, we conclude that all numbers from 1 to infinity are sums of 54 fifth powers.

* This limit was obtained by Dr. A. Oppenheim at the request of the writer.

Later I extended my table to 138,000. We shall use only the 39,400 numbers between 98,600 and 138,000. By the methods explained before, this part of the table enabled me to prove that all numbers between 98,605 and 200,000 are sums of 16 fifth powers. Using merely this fact, we may ascend as usual and find that all numbers having fewer than 194 figures are sums of 37 fifth powers, while all having fewer than 469 figures are sums of 41. We have not only secured better results than before, but also have secured an independent new proof of the main theorem that all numbers are sums of 54 fifth powers.

Finally, with the aid of assistants, I extended my table to 300,000. By using the minima $\geqq 14$ in the last half of the table, I proved that from 300,000 to 837,824 there are only 31 numbers of minimum 15 (and none of greater minimum), with an average of 1 per thousand in the first 15 thousands after 300,000, but with later thinning out and terminating with 470,348. Hence from the latter to 837,824, every number is a sum of 14 fifth powers. By the methods already explained, I used the last fact to prove that

All between	191,263 and	4,037,824 are sums of	15
	98,605	28,337,824	16
	87,919	310,813,073	17
	32,017	5,894,872,522	18
	30,221	222,594,738,147	19
	22,626	20,596 billion	20

. . . . . .

All with <484 figures are sums of 37
All with <1177 figures are sums of 41.

It therefore seems almost certain that every number is a sum of 37 fifth powers (while 37 are necessary to get the number 223). We also notice a dropping from 37 which is quite rapid to 18 and later quite slowly to 17, 16, 15, 14. A further dropping to 13 was secured recently. By studying the minima $\geqq 13$ just before 300,000, I proved that each of the 52,841 numbers between 786,159 (which requires 14 fifth powers) and 839,000 is a sum of 13 fifth powers. This remarkable fact implies that 14 suffice not merely to the former limit 837,824, but on to 940,000, which would permit us to raise somewhat all the limits in the last tablette.

A final word about the actual construction of my table of fifth powers. The earlier table of fourth powers was made by finding all ways of expressing each number as a sum of fourth powers and then picking out the ways involving the fewest powers. But the number of ways of expressing n as a sum of fourth (or fifth) powers increases extremely rapidly as n increases, whence this method is both highly impracticable and liable to many errors.

My method finds a minimum mode of expression without knowing all modes of expression. To illustrate, write

$$A = 2^5,\ B = 3^5,\ C = 4^5,\ D = 5^5,\ E = 6^5,\ F = 7^5,\ G = 8^5,$$

and consider decompositions involving only A, B, C, F. Those not appearing in the following tablette

$3F$	$B + 3F$	$2B + 3F$	$3B + 3F$
$A + 3F$	$A + B + 3F$	$A + 2B + 3F$	$A + 3B + 3F$
$2A + 3F$	$2A +$	$2A +$	
$3A + 3F$	$3A +$	$3A +$	
$4A + 3F$	$4A +$	$4A +$	
$5A + 3F$	$5A +$	$5A +$	

cannot be minimum decompositions in view of the four equations

$$6A + 3F = 2 + B + 2C + 2E + G,\ 2A + 3B + F = 2C + 2E,$$
$$4B + 3F = 1 + 3C + 2E + G,\ C + 3F = D + 2E + G.$$

The numerical values of the entries in our tablette were copied into the big table and the rare cases of duplicates and non-minima were deleted.

THE UNIVERSITY OF CHICAGO

AN EXISTENCE THEOREM FOR CHARACTERISTIC CONSTANTS OF KERNELS OF POSITIVE TYPE*

BY C. E. SEELY

We define a kernel $K(s, t)$ to be of positive type with respect to a set of functions $H(s)$ if

$$\int_a^b \int_a^b K(s, t)h(s)h(t)dsdt \geqq 0 \tag{1}$$

for every function $h(s)$ of the set $H(s)$.

Let the real kernel $K(s, t)$ be developable in a series of real normalized orthogonal functions $\phi_i(s)$, so that

$$K(s, t) = \sum_{i,j=1}^{\infty} a_{ij}\phi_i(s)\phi_j(t), \tag{2}$$

where

$$a_{ij} = \int_a^b \int_a^b K(s, t)\phi_i(s)\phi_j(t)dsdt, \tag{3}$$

and not all the a_{ij} are zero. We shall prove the following theorem.

THEOREM 1. *If $K(s, t)$ is of positive type with respect to the set of all functions of the form $c_\alpha\phi_\alpha(s)+c_\beta\phi_\beta(s)$, where the c's are real constants, then no coefficient a_{kk} is negative, and no a_{kk} is zero unless $a_{kj}+a_{jk}=0$ for every j.*

Suppose, for some subscript k, we have $a_{kk}<0$. Let $h(s)=\phi_k(s)$ in (1). Then we have, since the functions $\phi_j(s)$ form a normalized orthogonal set,

$$\begin{aligned} &\int_a^b \int_a^b K(s, t)\phi_k(s)\phi_k(t)dsdt \\ &= \int_a^b \int_a^b \sum_{i,j} a_{ij}\phi_i(s)\phi_j(t)\phi_k(s)\phi_k(t)dsdt \\ &= \int_a^b \sum_i a_{ik}\phi_i(s)\phi_k(s)ds = a_{kk} < 0, \end{aligned} \tag{4}$$

and we see that condition (1) is not satisfied.

* Presented to the Society, June 13, 1931.

Suppose now $a_{kk}=0$, and $a_{kj}+a_{jk}\neq 0$ for some j. Let us set $h(s)=c_k\phi_k(s)+c_j\phi_j(s)$ in condition (1), where the c's are arbitrary real constants. We have

$$\begin{aligned}
&\int_a^b\int_a^b K(s,t)[c_k\phi_k(s)+c_j\phi_j(s)][c_k\phi_k(t)+c_j\phi_j(t)]\,ds\,dt\\
&=\int_a^b\int_a^b\Big[\sum_{\alpha,\beta}a_{\alpha\beta}\phi_\alpha(s)\phi_\beta(t)\Big][c_k\phi_k(s)+c_j\phi_j(s)][c_k\phi_k(t)\\
&\quad+c_j\phi_j(t)]\,ds\,dt \qquad (5)\\
&=\int_a^b\Big[\sum_\alpha c_k a_{\alpha k}\phi_\alpha(s)+\sum_\alpha c_j a_{\alpha j}\phi_\alpha(s)\Big][c_k\phi_k(s)+c_j\phi_j(s)]\,ds\\
&=c_k^2 a_{kk}+c_jc_k a_{kj}+c_kc_j a_{jk}+c_j^2 a_{jj}.
\end{aligned}$$

Since $a_{kk}=0$, the right member of (5) reduces to

$$c_j^2 a_{jj}+c_jc_k(a_{kj}+a_{jk}). \qquad (6)$$

Since $a_{kj}+a_{jk}\neq 0$, we can choose c_j and c_k so that the quantity (6) is negative. This shows that a function $h(s)$ can be found for which condition (1) is not satisfied, which completes the proof of our theorem.

We now make the following additional assumptions for the kernel $K(s,t)$:

(i) $K(s,t)$ satisfies the continuity conditions required for applicability of the Fredholm theory.*

(ii) The series obtained by termwise integration of $\sum_{i,j=1}^{\infty}a_{ij}\phi_i(s)\phi_j(s)$ is convergent and represents $\int_a^b K(s,s)ds$.

(iii) The Fredholm determinant $D(\lambda)$ of $K(s,t)$ is of genus zero.

A sufficient condition for (iii) is that

$$\left|\frac{K(s,t_1)-K(s,t)}{t_1-t}\right|<M$$

for some constant M, independent of s, t, and t_1.†

From (i) we have the well known development

* See, for example, Goursat, *Cours d'Analyse*, vol. 3, p. 342.

† See Fredholm, Acta Mathematica, vol. 27, p. 368.

$$(7)\quad D(\lambda) = 1 - \lambda \int_a^b K(s, s)ds + \cdots + \frac{(-\lambda)^p}{p!} \int_a^b \cdots \int_a^b K\begin{pmatrix} s_1 \cdots s_p \\ s_1 \cdots s_p \end{pmatrix} ds_1 \cdots ds_p + \cdots,$$

and from (iii)*

$$(8)\quad D(\lambda) = \prod_{i=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_i}\right) = 1 - \lambda\left(\sum \frac{1}{\lambda_i}\right) + \cdots,$$

where the λ_i are the characteristic constants of the kernel.

Equating coefficients of λ in (7) and (8) and applying assumption (ii), we have

$$(9)\quad \int_a^b K(s, s)ds = \sum \frac{1}{\lambda_i} = \sum a_{ii}.$$

It is clear that, under our present assumptions, $K(s, t)$ must have at least one finite characteristic constant if $\sum a_{ii} \neq 0$.

Now $\sum a_{ii}$ cannot vanish unless either some a_{kk} is negative or every a_{kk} is zero. From Theorem 1, $K(s, t)$ cannot be of positive type with respect to the set of all functions of the form $c_\alpha\phi_\alpha(s) + c_\beta\phi_\beta(s)$ if any a_{kk} is negative, nor if any a_{kk} vanishes unless $a_{kj} + a_{jk} = 0$ for every j. If every a_{kj} and a_{jk} vanishes, we have the trivial case where $K(s, t)$ vanishes identically, which has been excluded. But if every a_{kk} is zero and for every k and j, $a_{kj} = -a_{jk}$, $K(s, t)$ is skew-symmetric: $K(s, t) = -K(t, s)$, and hence has at least one characteristic constant.†

We have therefore proved the following theorem.

THEOREM 2. *If a given kernel $K(s, t)$, developable in a series $\sum a_i\phi_i(s)\phi_i(t)$ of normalized orthogonal functions, is of positive type with respect to the set of all functions of the form $c_\alpha\phi_\alpha(s) + c_\beta\phi_\beta(s)$, and satisfies the conditions* (i), (ii), *and* (iii), *it has at least one finite characteristic constant.*

It will be noted that the conditions (i), (ii), and (iii) all refer to continuity properties rather than to symmetry.

NEW YORK CITY

* See, for example, Goursat, *Cours d'Analyse*, vol. 2, pp. 152–153; vol. 3, pp. 425–426.

† See, for example, Goursat, *Cours d'Analyse*, vol. 3, p. 468.

SOME THEOREMS ON PLANE CURVES

BY W. V. PARKER

In applying Abel's theorem to hyperelliptic integrals, we are interested in the intersections of certain curves with a curve C of the type $y^2=f(x)$, where $f(x)$ is a polynomial. The functions used in the following are all polynomials of degree indicated by their subscripts. If $f_n(x)\equiv f_k(x)f_{n-k}(x)$ we may without any loss of generality assume that $n\geqq k\geqq n/2$ and this assumption will be made throughout.

LEMMA. *If C is the curve $y^2=f_n(x)\equiv f_k(x)f_{n-k}(x)$, c_1 the curve $y=f_k(x)$ and c_2 the curve $y=f_{n-k}(x)$, then all the finite points of intersection of c_1 and c_2 are on C, and the curve S whose equation is $y=[f_k(x)+f_{n-k}(x)]/2$ is tangent to C at each of these k points.*

Suppose (α, β) is any one of the k points of intersection of c_1 and c_2; then $\beta=f_k(\alpha)$ and $\beta=f_{n-k}(\alpha)$ and therefore $\beta^2=f_k(\alpha)f_{n-k}(\alpha)=f_n(\alpha)$, that is (α, β) is on C. Obviously S passes through the k points of interesection of c_1 and c_2 and hence meets C in these k points. Eliminating y from the equations of S and C we get

$$\left[\frac{f_k(x)+f_{n-k}(x)}{2}\right]^2-f_k(x)f_{n-k}(x)\equiv\left[\frac{f_k(x)-f_{n-k}(x)}{2}\right]^2=0$$

as the equation giving the abscissas of the $2k$ points of intersection of S and C. Since the left hand side of this equation is a perfect square each abscissa is counted twice, and therefore since, in S, y is a one-valued function of x, S is tangent to C at each of these k points.

As an immediate consequence of this lemma we have the following result.

THEOREM 1. *If C is the curve $y^2=\phi_n(x)$, where $\phi_n(e_i)=0$, $(i=1, \cdots, n)$, and (α, β), $(\beta\neq 0)$, is a point on C, and c_1 is the curve of the form $y=\phi_k(x)$ determined by (α, β) and any k of the points $(e_i, 0)$, and c_2 is the curve of the form $y=\phi_{n-k}(x)$ determined by (α, β) and the remaining $n-k$ of the points $(e_i, 0)$, then c_1 and*

c_2 have all their k points of intersection on C, and the curve S whose equation is $y=[\phi_k(x)+\phi_{n-k}(x)]/2$ is tangent to C at each of these k points.*

Since $\phi_n(x)=\phi_k(x)\phi_{n-k}(x)$ for $n+1$ values of x, we have $\phi_n(x)\equiv\phi_k(x)\phi_{n-k}(x)$ and the theorem follows from the lemma.

That all curves S of the form $y=g_k(x)$ which are tangent to a curve C of the form $y^2=g_n(x)$ at each of k points can be obtained by this process, is a consequence of the following theorem.

THEOREM 2. *If (α_i, β_i), $(i=1, 2, \cdots, k)$, are k points on the curve C whose equation is $y^2=g_n(x)$ such that there exists a curve S of the form $y=g_k(x)$ which is tangent to C at each of these k points, and if the curve c_1 whose equation is $y=h_k(x)$ meets C in the k points (α_i, β_i) and the point $(e_\lambda, 0)$, where e_λ is any zero of $g_n(x)$, then $h_k(x)$ is a factor of $g_n(x)$.*

Since S is tangent to C at each of the k points, the equation $g_k^2(x)-g_n(x)=0$ has the roots $\alpha_1, \alpha_2, \cdots, \alpha_k$, each counted twice, and since c_1 meets S in the k points (α_i, β_i), the equation $g_k(x)-h_k(x)=0$ has the roots $\alpha_1, \alpha_2, \cdots, \alpha_k$.

We have therefore

$$[g_k(x) - h_k(x)]^2 \equiv \mu[g_k^2(x) - g_n(x)],$$

and hence

$$[g_k(e_\lambda) - h_k(e_\lambda)]^2 = \mu[g_k^2(e_\lambda) - g_n(e_\lambda)];$$

but $h_k(e_\lambda)=g_n(e_\lambda)=0$, hence $\mu=1$, and we have

$$g_k^2(x) - 2g_k(x)h_k(x) + h_k^2(x) \equiv g_k^2(x) - g_n(x),$$

or

$$g_n(x) \equiv h_k(x)[2g_k(x) - h_k(x)].$$

If c_1 is the curve $y=a_0x^k+a_1x^{k-1}+\cdots+a_{k-1}x+a_k$ determined by the k points (α_i, β_i) and one of the n points $(e_i, 0)$, the coefficient a_0 may be zero and the degree of the right hand side less than k. For suppose we choose a particular one, say e_1, of

* Only finite points of intersection are considered here. In certain special cases when n is even and $k=\frac{1}{2}n$, c_1 and c_2 may coincide or they may have less than k finite points of intersection. The lemma and Theorem 1 are still true for these cases when finite points of intersection are considered.

the zeros of $g_n(x)$ and find that the expression on the right is of degree k; then it will have as zeros k of the zeros of $g_n(x)$, say $e_1, e_2, \cdots, e_k$. Then the curve $y=b_0x^k+b_1x^{k-1}+\cdots+b_{k-1}x+b_k$ determined by the k points (α_i, β_i) and one of the remaining points $(e_i, 0)$, say $(e_{k+1}, 0)$, will have its right hand side of degree $n-k$ at most. For suppose the right hand side of degree $m>n-k$; then it will have as zeros m of the zeros of $g_n(x)$ and and hence at least one of the $e_1, e_2, \cdots, e_k$ and therefore $a_0x^k+a_1x^{k-1}+\cdots+a_k=b_0x^k+b_1x^{k-1}+\cdots+b_k$ for at least $k+1$ values. But since $b_0x^k+b_1x^{k-1}+\cdots+b_k$ has at least one zero which is not a zero of $a_0x^k+a_1x^{k-1}+\cdots+a_k$ this is impossible. It follows as a consequence of Theorem 1 that the degree of the right hand side is either k or $n-k$ depending on which zero of $g_n(x)$ is chosen for determining the curve c_1.

If in the above the degree of $h_k(x)$ is k, the degree of $2g_k(x)-h_k(x)$ will be $n-k$; if we denote the latter by $h_{n-k}(x)$, we shall have $g_k(x)\equiv[h_k(x)+h_{n-k}(x)]/2$. That is, the curve S is $y=[h_k(x)+h_{n-k}(x)]/2$, where the curve $y=h_k(x)$ is determined by some k of the points $(e_i, 0)$ and one of the points (α_i, β_i), and the curve $y=h_{n-k}(x)$ is determined by the remaining $n-k$ of the points $(e_i, 0)$ and the same one of the points (α_i, β_i).

Thus far it has not been necessary to say anything about the nature of the zeros $e_1, e_2, \cdots, e_n$. When these zeros are distinct we have the following theorem.

THEOREM 3. *The number of curves of the type $y=g_k(x)$ which are tangent to a curve C of the type $y^2=g_n(x)$ at any fixed point (α, β) and at $k-1$ other points, is C_k^n for $k>n/2$ and $\frac{1}{2}C_k^n$ for $k=n/2$, provided that the zeros of $g_n(x)$ are distinct.*

For by Theorem 1 we get a curve of this type corresponding to any k of the zeros of $g_n(x)$ and by Theorem 2 all curves of this type are obtained by this process. It must be shown, therefore, that when $k>n/2$ the same curve cannot be obtained from two different sets of k zeros of $g_n(x)$. Suppose $y=\phi_k(x)$ and $y=\psi_k(x)$ are both of degree k and cut out the same set of k points (α_i, β_i) on C; then $\phi_k(x)$ and $\psi_k(x)$ must have at least one zero in common and therefore $\phi_k(x)\equiv\psi_k(x)$. If n is even and $k=\frac{1}{2}n$, then each set of k such points is cut out by two and only two of these curves by Theorem 1.

From Theorem 1, the ordinary construction for drawing a

tangent to a conic at a point P on it, when the axes and vertices are known, follows immediately.

The following example is a rather interesting illustration of Theorem 1. Let C be the curve

$$y^2 = f_6(x) \equiv -x^6 + 14x^4 - 49x^2 + 36.$$

The zeros of $f_6(x)$ are $1, -1, 2, -2, 3, -3$. Let the curve $c_1: y = f_3(x)$ be determined by $(0, 6)$ $(1, 0)$ $(-1, 0)(3, 0)$ and the curve $c_2: y = g_3(x)$ be determined by $(0, 6)$ $(2, 0)$ $(-2, 0)$ $(-3, 0)$; then we have

$$f_3(x) \equiv 2x^3 - 6x^2 - 2x + 6,$$
$$g_3(x) \equiv -\tfrac{1}{2}x^3 - \tfrac{3}{2}x^2 + 2x + 6.$$

These curves c_1 and c_2 meet on C in three points whose abscissas are 0, $(9+\sqrt{241})/10$, $(9-\sqrt{241})/10$. The curve S whose equation is

$$y = \frac{f_3(x) + g_3(x)}{2} \equiv \frac{3}{4}x^3 - \frac{15}{4}x^2 + 6$$

is tangent to C at each of these three points.

If we take for c_1 the curve $y = g_0(x)$ determined by $(0, 6)$, and for c_2 the curve $y = g_6(x)$ determined by $(0, 6)$ $(1, 0)$ $(-1, 0)$ $(2, 0)$ $(-2, 0)$ $(3, 0)$ $(-3, 0)$, we get

$$g_0(x) \equiv 6, \qquad g_6(x) \equiv -\frac{1}{6}x^6 + \frac{7}{3}x^4 - \frac{49}{6}x^2 + 6.$$

The curves c_1 and c_2 are each tangent to C at each of the three points $(0, 6)$ $(\sqrt{7}, 6)$ $(-\sqrt{7}, 6)$ and the curve S whose equation is

$$y = \frac{g_0(x) + g_6(x)}{2} = -\frac{1}{12}x^6 + \frac{7}{6}x^4 - \frac{49}{12}x^2 + 6$$

meets C four times at each of the three points.

THE UNIVERSITY OF NORTH CAROLINA

A PROOF OF THE IDENTITY OF THE RIESZ INTEGRAL AND THE LEBESGUE INTEGRAL*

BY W. M. WHYBURN

1. *Introduction.* In Acta Mathematica, volume 42, pages 191–205, Friedrich Riesz developed a theory of integration independent of the theory of measure of point sets except in so far as sets of measure zero were involved. His theory yields an integral which he showed to have many of the properties of the Lebesgue integral and which he showed to exist and to be identical with the Lebesgue integral when this latter integral exists. Riesz' treatment yields a theory of measure which has the essential characteristics of the Lebesgue theory. Riesz does not seem to have proved that his development yields an integral which is identical with the Lebesgue integral† and that his notion of measure is the same as the ordinary notion. The present paper does show the entire identity of the Riesz integral and the Lebesgue integral. The terminology and notation of Riesz' paper are used.

THEOREM. *A necessary and sufficient condition that a bounded function on* $X: a \leqq x \leqq b$ *be measurable is that there exist a sequence of simple functions‡ which approaches this function almost everywhere on* X.

2. *Proof of Necessity.*§ Let $f(x)$ be bounded and measurable on X and let a method of subdivision of X be chosen in such a way that the length of the longest subdivision approaches zero as the number of these subdivisions becomes infinite. $f(x)$ is Lebesgue integrable on X. Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be the subdivision points at the nth stage and let

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* Presented to the Society, April 11, 1931.

† At the bottom of page 199 of Riesz' paper one finds a statement which indicates that he suspected a close relationship between or even the identity of, the two integrals; but he makes no positive statement, and gives no proof.

‡ Also called *step functions* or *horizontal functions*.

§ This part of the theorem is well known. A proof is included here for completeness.

$$\phi_n(x) = \int_{x_{i-1}}^{x_i} \frac{f(t)dt}{x_i - x_{i-1}}$$

on $x_{i-1} < x < x_i$, $(i = 1, 2, \cdots, n)$. Since $F(x) = \int_a^x f(t)dt$ is absolutely continuous on X, its derivative exists and is equal to $f(x)$ almost everywhere on X. Let A be the point set of zero measure that is made up of all points of X for which $F'(x)$ fails to exist together with all subdivision points. Let $x = p$ be any point of $X - A$ and let the notation be chosen so that $x_i < p < x_{i+1}$ for each n. We have

$$\phi_n(p) = \int_{x_i}^{x_{i+1}} \frac{f(x)dx}{x_{i+1} - x_i} = \int_{x_i}^{p} \frac{f(x)dx}{p - x_i} + \left[\int_{p}^{x_{i+1}} \frac{f(x)dx}{x_{i+1} - p} - \int_{x_i}^{p} \frac{f(x)dx}{p - x_i} \right] \frac{x_{i+1} - p}{x_{i+1} - x_i}. \tag{1}$$

Since $(x_{i+1} - p)/(x_{i+1} - x_i) < 1$ and each of the terms in the bracket of (1) approaches $F'(p) = f(p)$, it follows that the second term of the right hand side of equation (1) approaches zero when n becomes infinite. The first term on the right hand side of (1) has $F'(p) = f(p)$ for its limit as n becomes infinite (since $x_i \to p$) and hence $\lim_{n\to\infty} \phi_n(p) = f(p)$. We have therefore shown that $\{\phi_n(x)\}$ approaches $f(x)$ almost everywhere on X. We also note that the area under $\phi_n(x)$ is $\int_a^b f(x)dx$ for each n.

3. *Proof of Sufficiency.* We give two proofs of this part of the theorem. Let $\{\phi_n(x)\}$ be a uniformly bounded sequence of simple functions approaching $f(x)$ almost everywhere on X (such a sequence can be constructed from the set given in the theorem by substituting the bounds of $f(x)$ for portions of functions which exceed these bounds in numerical value). By Egeroff's theorem* $\{\phi_n(x)\}$ approaches $f(x)$ uniformly on X except for a set of points of arbitrarily small outer measure. Let M be any constant. We show that the sets on which $f(x) > M$, $f(x) < M$, and $f(x) = M$ are measurable. Let ϵ be an arbitrarily assigned positive number and let N_i be chosen so that for all $n \geqq N_i$ and for all x on X except for points of a set of outer measure less than $\epsilon/2^{i+1}$, $|f(x) - \phi_n(x)| < 1/2^i$. Let G_i be the

* See Hobson, *Functions of a Real Variable*, 1926, vol. 2, p. 140.

finite set of intervals on which $\phi_{N_i}(x) > M + 1/2^i$. With the possible exception of a set of points of outer measure less than $\epsilon/2^{i+1}$, $f(x) > M$ on G_i. The set $\sum G_i$ that is obtained when i takes on all positive integral values may be replaced by an at most countably infinite set g of non-overlapping intervals which has the same interior. Let L be the sum of the lengths of the intervals of g and note that g contains all points of X for which $f(x) > M$ except for points of the null set on which $\{\phi_n(x)\}$ does not converge to $f(x)$. The subset of g on which $f(x) \leqq M$ is of outer measure less than $\sum_1^\infty \epsilon/2^{i+1} < \epsilon$ and hence the inner measure of the set on which $f(x) > M$ differs from L by less than ϵ. The outer measure of the set on which $f(x) > M$ cannot exceed L since g is an admissible covering of this set. The set on which $f(x) > M$ is therefore measurable since its inner and outer measures differ by an arbitrary number ϵ. Similar reasoning shows that the set on which $f(x) < M$ is measurable. The set on which $f(x) = M$ is measurable, since it is the difference between the measurable set X and the sum of the two sets whose measurabilities have just been demonstrated. Hence $f(x)$ is measurable on X.

4. *Second Proof of Sufficiency.* We use the Baire classification of functions.* Let A be the set of all points of X at which any function of $\{\phi_n(x)\}$ is discontinuous and let B be the set of points at which $\{\phi_n(x)\}$ does not approach $f(x)$. The set $C = A + B$ is of measure zero and the sets $X - A$ and $X - C$ are dense in themselves. For each n, $\phi_n(x)$ is of the first class† on $X - A$. Hence $f(x)$ is of the second class on $X - C$. It follows that $f(x)$ is measurable‡ on $X - C$ and is therefore measurable on X.

The foregoing theorem shows that the class of bounded functions that are integrable in the Riesz sense is identical with the class of bounded functions that are integrable in the Lebesgue sense. Riesz' results show that the two integrals are equal when they both exist, and we have shown in this paper that when either exists, the other does. The complete equivalence of the two integrals for bounded functions is therefore established. In the presence of this equivalence one needs but to examine the

* See Carathéodory, *Vorlesungen ueber reelle Funktionen*, 1927, pp. 393–413.
† See Carathéodory, loc. cit., page 403, Satz 3.
‡ See Carathéodory, loc. cit., page 403, Satz 4.

definitions of Riesz and Lebesgue for integrals of unbounded functions to see that the equivalence extends to such functions. We may therefore state that *the class of functions summable in the Lebesgue sense is identical with the class of functions summable in the Riesz sense and the two theories lead to one and the same integral.* As a corollary to this result we get the equivalence of the Riesz and Lebesgue theories of measure.

In connection with unbounded functions it might be observed that one could define the integral of such a function so that it would exist when either the Lebesgue integral or the improper Riemann integral exists.* Such a definition would completely remove the necessity of separate treatments of Lebesgue and Riemann integration as it, together with the results of the present paper, would permit both theories to be presented simultaneously from the Riesz point of view. The following two theorems emphasize this point of view.

A necessary and sufficient condition that a bounded function be Lebesgue integrable on $X: a \leqq x \leqq b$ *is that it be the limit function almost everywhere of a sequence* $\{\phi_n(x)\}$ *of simple functions.*

The present paper establishes this theorem. Riesz† proves the following second theorem.

A necessary and sufficient condition that a bounded Lebesgue integrable function be Riemann integrable is that the sequence $\{\phi_n(x)\}$ *approach the function uniformly‡ almost everywhere on* X.

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* Since the present paper was written, I have used the Riesz point of view to define an integral of an unbounded function which has this property. This integral is unique and it may exist when neither the Lebesgue nor the Riemann integral exists. I have also established a necessary and sufficient condition (in terms of simple functions) that a function (which may be unbounded) be summable in the Lebesgue sense. It is my intention to publish these results in the near future.

† Loc. cit., p. 204.

‡ Riesz, loc. cit., p. 204, defines *uniform approach almost everywhere* on X to mean uniform approach in the neighborhoods of all points of X with the possible exception of a set of measure zero.

INVERSE TERNARY CONTINUED FRACTIONS

BY D. N. LEHMER

In Jacobi's extension of the continued fraction algorithm* we are concerned with three series of numbers given by the recursion formulas

$$A_n = q_n A_{n-1} + p_n A_{n-2} + A_{n-3},$$
$$B_n = q_n B_{n-1} + p_n B_{n-2} + B_{n-3},$$
$$C_n = q_n C_{n-1} + p_n C_{n-2} + C_{n-3},$$

with initial values 1, 0, 0 for A; 0, 1, 0 for B, and 0, 0, 1 for C. We have called this series of numbers (A_n, B_n, C_n) the *convergent sets*, and the series of numbers (p_n, q_n) the *partial quotient sets* of a *ternary continued fraction*.† It is well known that if the partial quotient sets recur periodically the ratios A_n/B_n, A_n/C_n and B_n/C_n approach cubic irrationalities except in certain special cases where they approach quadratic irrationalities‡ or where they approach no limit at all. The cubic irrationalities when they exist are connected by a linear fractional relation with the roots of the *characteristic cubic*

$$\begin{vmatrix} A_{k-2} - \rho & B_{k-2} & C_{k-2} \\ A_{k-1} & B_{k-1} - \rho & C_{k-1} \\ A_k & B_k & C_k - \rho \end{vmatrix} = 0,$$

connected with the *purely periodic* ternary continued fraction $(p_1, q_1; p_2, q_2; \cdots; p_k, q_k)$ formed of those partial quotient pairs that recur. This characteristic cubic we write $\rho^3 - M\rho^2 + N\rho - 1 = 0$, where $M = A_{k-2} + B_{k-1} + C_k$, and

$$N = A_{k-2}B_{k-1} - A_{k-1}B_{k-2} + B_{k-1}C_k - B_kC_{k-1} + C_kA_{k-2} - A_kC_{k-2}.$$

We shall confine ourselves in what follows to purely periodic

* Jacobi, Werke, vol. VI, pp. 385–426.

† Proceedings of the National Academy of Sciences, vol. 4 (1918), p. 360.

‡ J. B. Coleman, American Journal of Mathematics, vol. 52, No. 4, October, 1930.

fractions. Those fractions which have a finite number of non-periodic partial quotient pairs to start with introduce no new difficulties. The irrationalities which they determine are simply related to the fraction obtained by ignoring the non-periodic pairs.

Two ternary continued fractions $(p_1, q_1; p_2, q_2; \cdots; p_n, q_n)$ and $(p_n, q_n; p_{n-1}, q_{n-1}; \cdots; p_2, q_2; p_1, q_1)$ will be called *inverse*. If a purely periodic ternary continued fraction be set up with the first of these as its period and another with the second as its period the two continued fractions will not necessarily have the same discriminating cubic, unless the partial quotient pairs read backward and forward the same. Thus the cubic which corresponds to the fraction $(1, 1; 3, 2; 4, 1)$ is $\rho^3-14\rho^2-1=0$ and that corresponding to the fraction $(4, 1; 3, 2; 1, 1)$ is $\rho^3-17\rho^2+3\rho-1=0$. The irrationalities defined by these cubics are different. It is the purpose of this paper to show that if the p's and q's are given by the equations

$$p = \alpha t + \beta, \quad q = \gamma t + \delta, \tag{1}$$

where α, β, γ, δ, t are any given integers whatever, positive, negative or zero, then the discriminating cubic is the same for the fraction and its inverse. This theorem we establish by induction, but in the treatment by this method it appears that it is necessary to drive along simultaneously a group of assumed relations. This peculiarity, seldom met with elsewhere, seems to be characteristic of the proofs by induction of theorems in continued fractions of higher orders.

Let the nth convergent set of the first fraction be (A_n, B_n, C_n) and of the second be (A_n', B_n', C_n'), and let the coefficients of the discriminating cubic of the first be M and N and of the second M' and N'. We have to show that if the equations (1) hold, then $M=M'$ and $N=N'$. These equations are easily verified, for $n=1, 2, \cdots$. We assume them true for values of the index up to and including n. We consider then the fraction with $n+1$ partial quotient pairs: $(p_1, q_1; p_2, q_2; \cdots; p_n, q_n; p, q)$ where p and q also satisfy equations (1) and let the nth convergent set of the inverse of this fraction be (A_n'', B_n'', C_n''). By the formula given on page 361 of my note on Jacobi's algorithm (loc. cit.), we have for the last three convergent sets of this fraction:

$$(C'_{n-2},\ pC'_{n-2}+A'_{n-2},\ qC'_{n-2}+B'_{n-2}),$$
$$(C'_{n-1},\ pC'_{n-1}+A'_{n-1},\ qC'_{n-1}+B'_{n-1}),$$
$$(C'_{n},\ \ pC'_{n}+A'_{n},\ \ qC'_{n}+B'_{n}).$$

From the hypothesis that $M=M'$ we have

$$A_{n-2}+B_{n-1}+C_n = A'_{n-2}+B'_{n-1}+C'_n, \tag{2}$$

and we wish to show that

$$A_{n-1}+B_n+qC_n+pC_{n-1}+C_{n-2} = A''_{n-1}+B''_n+C''_{n+1}$$
$$= C'_{n-2}+pC'_{n-1}+A'_{n-1}+qC'_n+B'_n.$$

Replacing p and q by their values from (1), we have

$$A_{n-1}+B_n+C_{n-2}+(\gamma t+\delta)C_n+(\alpha t+\beta)C_{n-1}$$
$$= A'_{n-1}+B'_n+C'_{n-2}+(\gamma t+\delta)C'_n+(\alpha t+\beta)C'_{n-1}. \tag{3}$$

If this equation is to hold for all values of t, we should have, equating coefficients of t,

$$\gamma C_n+\alpha C_{n-1} = \gamma C'_n+\alpha C'_{n-1}. \tag{4}$$

We now assume this equation to hold and proceed as before to $n+1$. In the resulting equation the coefficient of t vanishes by virtue of equation (4). The terms independent of t must satisfy the equation

$$\alpha C_n+\gamma(\delta C_n+\beta C_{n-1}+C_{n-2}) = \gamma\delta C'_n+\alpha\delta C'_{n-1}+\gamma B'_n+\alpha B'_{n-1}. \tag{5}$$

This equation we also assume to be true for the values of n in question, and on proceeding to $n+1$ we find the coefficients of t vanish by virtue of (5). The remaining terms reduce by (4) and (5) to the equation

$$\gamma C_{n-1}+\alpha C_{n-2} = \gamma A'_n+\alpha A'_{n-1}. \tag{6}$$

On assuming this equation to hold and proceeding to the value $n+1$ we obtain equation (4) again. Equations (4), (5) and (6) thus form a closed system. We return to equation (3) and equate the parts independent of t. The resulting equation,

$$A_{n-1}+B_n+C_{n-2}+\delta C_n+\beta C_{n-1}$$
$$= A'_{n-1}+B'_n+C'_{n-2}+\delta C'_n+\beta C'_{n-1}, \tag{7}$$

we also assume to hold for all values of the index up to and including n. Proceeding to $n+1$ we find that the coefficient of t vanishes by virtue of (5), while the terms independent of t yield the equation

$$(8)\quad A_n+\delta B_n+\beta B_{n-1}+B_{n-2}+C_{n-1}+\delta^2 C_n+\beta\delta C_{n-1}+\delta C_{n-2}+\beta C_n$$
$$=A_n'+\delta B_n'+\beta B_{n-1}'+B_{n-2}'+C_{n-1}'+\delta^2 C_n'+\beta\delta C_{n-1}'+\delta C_{n-1}'+\beta C_n'.$$

If we assume this equation and proceed to $n+1$, we find that the resulting equations are found to be true by virtue of the preceding equations. Equations (4), (5), (6), (7), and (8) with equation (2) form then a closed system of equations, all of which may be verified for $n=1, 2, 3, \cdots$. Assuming them true for values of the index up to n, they therefore hold for values up to $n+1$. They therefore hold in general.

The corresponding discussion for the assumed equation $N=N'$ leads to the following set of equations, where for shortness we write (A_i, B_j) for $A_iB_j-A_jB_i$:

$$(9)\quad (A_{n-1}, B_n)+(A_{n-2}, C_n)+(B_{n-1}, C_n)$$
$$=(A_{n-1}', B_n')+(A_{n-2}', C_n')+(B_{n-1}', C_n').$$

$$(10)\quad \gamma(A_{n-1}, C_n)+\alpha(B_n, C_{n-1})=\gamma(A_{n-1}', C_n')+\alpha(B_n', C_{n-1}').$$

$$(11)\quad \gamma(A_n, C_{n-2})+\alpha(C_n, B_{n-2})$$
$$=(\alpha\delta-\beta\gamma)(A_n', C_{n-1}')+\gamma(C_{n-1}', B_n')+\alpha(A_n', B_{n-1}').$$

$$(12)\quad \gamma(A_{n-2}, C_{n-1})+\alpha(C_{n-2}, B_{n-1})$$
$$=\gamma(A_n', B_{n-1}')+\alpha(A_{n-1}', C_n').$$

$$(13)\quad (A_{n-1}, B_n)+(A_{n-1}, C_{n-2})+(B_n, C_{n-2})+\delta(A_{n-1}, C_n)$$
$$+\beta(B_n, C_{n-1})=(A_{n-1}', B_n')+(A_{n-1}', C_{n-2}')+(B_n', C_{n-2}')$$
$$+\delta(A_{n-1}', C_n')+\beta(B_n', C_{n-1}').$$

$$(14)\quad (A_n, B_{n-2})+(A_n, C_{n-1})+\delta(B_n, C_{n-1})+(B_{n-2}, C_{n-1})$$
$$+\delta(A_n, C_{n-2})+\beta(A_{n-1}, C_{n-2})=(A_n', B_{n-2}')+(A_n', C_{n-1}')$$
$$+\delta(B_n', C_{n-1}')+(B_{n-2}', C_{n-1}')+\delta(A_n', C_{n-2}')$$
$$+\beta(A_{n-1}', C_{n-2}').$$

This is found to be a closed system of equations, and the assumption of any one is justified by the others. A simple trial verifies them for the values of $n=1, 2, 3$. They are then true in general. We have then the following theorem.

THEOREM 1. *In any periodic, ternary continued fraction the discriminating cubic is not changed when the order of the partial quotient pairs is inverted provided that the partial quotient pairs satisfy the equations*

$$p_i = \alpha t + \beta, \quad q_i = \gamma t + \delta,$$

where α, β, γ and δ are any fixed integers, positive, negative or zero, and the parameter t also takes any integer values positive, negative or zero.

As an interesting special case we may take α or γ equal to zero. Taking $\alpha = \delta = 0$, $\gamma = 1$, we get the following theorem.

THEOREM 2. *The discriminating cubic is not changed by inversion of order of the partial quotient sets in any periodic ternary continued fraction in which p is fixed while q takes any values, or q is fixed while p takes any values.*

Such continued fractions might be called *linear*, since the equation between p and q requires them to be the coordinates of points on a line. With some difficulty, due to the complicated algebra involved, we have shown that if in a quaternary continued fraction with periodic partial quotient sets (p, q, r) we have the equations

$$p_i = \alpha t + \beta, \quad q_i = \gamma t + \delta, \quad r_i' = \lambda t + \mu,$$

where $\alpha, \beta, \gamma, \delta, \lambda, \mu$ are fixed integers, positive, negative or zero, while the parameter t takes any integer values, positive, negative or zero, then the order of the partial quotient sets may be inverted without changing the discriminating quartic. A similar theorem seems to hold for fractions of higher order, but a general proof is lacking.

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ON COMPLETE SYSTEMS UNDER CERTAIN FINITE GROUPS

BY C. W. STROM

1. *Introduction.* The theory of complete systems of invariants for rational functions that are unaltered under the substitutions of a finite group finds a close parallel in the theory of the binary n-ic.

It is well known that all the invariants and covariants of a binary n-ic can be expressed as rational functions of just n explicitly known forms such that the denominators are powers of the n-ic itself. In the parallel case Lagrange's theorem states that every rational function $F(x_1, \cdots, x_n)$ that is unaltered under the substitutions of a finite group G on $x_1, \cdots, x_n$ can be expressed as a rational function of the elementary symmetric polynomials $E_i, (i=1, \cdots, n)$, and any particular F that belongs to G. In these representations the denominator is a particular symmetric polynomial depending only on F.

If complete integrality is insisted upon, it is well known that the number of members in the complete system of the binary n-ic is finite although the exact number and the explicit forms cannot in general be determined by known means. In this paper we study the problem of obtaining irreducible sets of polynomials such that all polynomials that are invariant under the substitutions of a certain finite group can be expressed as rational, integral functions of the members of the irreducible set. In the case of the symmetric group $G^n_{n!}$ the answer is given by the fundamental theorem of symmetric functions. The results for the alternating group $G^n_{n!/2}$ and the identity group G_1 are also well known. We present solutions for the cyclic groups, the solvable groups, and the simple group G^7_{168}. In the cases of the cyclic groups and the solvable groups, the number of members in the irreducible sets is shown to be finite but the determination of the exact number is resolved essentially into a problem of partitions and can therefore not be expressed in general by known means.

If we regard $x_1, \cdots, x_n$ as coordinates in S_{n-1}, we may regard the group G as a collineation group and the results may be in-

terpreted as giving complete systems of invariants for certain collineation groups.

2. *The Cyclic Groups.* We consider a cyclic substitution of order n:

$$s = (x_0 x_1 \cdots x_{n-1}).$$

If

$$y_i = x_0 + \epsilon^i x_1 + \cdots + \epsilon^{(n-1)i} x_{n-1}, \quad \epsilon = e^{2\pi i/n},$$

$$(i = 0, 1, \cdots, (n-1)),$$

then

$$n x_i = y_0 + \epsilon^{(n-i)} y_1 + \cdots + \epsilon^{(n-1)(n-i)} y_{n-1}.$$

Hence, if $P(x)$ is any polynomial in $x_0, x_1, \cdots, x_{n-1}$, we have

$$P(x) = \Pi(y) = \sum_{\alpha_0, \cdots, \alpha_{n-1}} A_{\alpha_0, \cdots, \alpha_{n-1}} y_0^{\alpha_0} \cdots y_{n-1}^{\alpha_{n-1}}.$$

Since $s(y_i) = \epsilon^i y_i$, we have the following theorem.

THEOREM 1. *A necessary and sufficient condition that $P(x)$ be invariant under the substitutions of the cyclic group G_n^n is that it be expressible as the sum of products of arbitrary powers of y_0 and products like $y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}$, where $\alpha_1, \cdots, \alpha_{n-1}$ form a positive integral solution of the diophantine equation*

$$\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1} = rn, \qquad (r = 1, 2, 3, \cdots).$$

The solution of this equation is easy in particular cases and is dependent in general on partitions. Any solution of this equation yields a polynomial that is invariant under the given substitution. A finite set of polynomials to form a complete system is obtained from the aggregate of solutions by elementary considerations.

The determination of a complete system for the polynomials that belong to the cyclic group G_n^{qn} requires q sets of cogredient y_i's like those above and the solution of the diophantine equation

$$\sum_{j=1}^{q} [\alpha_1^{(j)} + 2\alpha_2^{(j)} + \cdots + (n-1)\alpha_{n-1}^{(j)}] = rn, \quad (r = 1, 2, 3, \cdots).$$

3. *The Solvable Groups.* If a polynomial P belongs to a subgroup H of index n under a finite group G, then P is n-valued under G. If H is self-conjugate under G, then the conjugates of P are permuted under G according to the factor group G/H. If the complete system of H is known consisting of q polynomials, and if the factor group G/H is cyclic, we have q sets of n quantities each permuted according to the cyclic group G_n^{qn}. A complete system for G can then be found by the methods outlined above. A repetition of this process enables us to obtain a complete system of invariants for the polynomials that belong to any composite group G whose series of composition $G, G_1, \cdots, G_\mu, 1$ is such that the factor groups $G_\mu, G_{\mu-1}/G_\mu, \cdots, G/G_1$ are all cyclic. If we assume, as we may without loss of generality, that each of the subgroups in the series of composition of G is a maximal self-conjugate subgroup, then the groups defined are just the solvable groups. For if the self-conjugate subgroups are all maximal, the factor groups are all simple. If the factor groups are also cyclic they must be of prime order and G is a solvable group.

4. *The Simple Group* G_{168}^7. The elementary symmetric polynomials on seven letters, E_i, $(i=1, \cdots, 7)$, are clearly invariant under this group. However, E_3 and E_4 break up into parts that are themselves invariant under the group. We shall call these parts *S-polynomials*. They play a role in this theorem analogous to that of the E_i in the fundamental theorem of symmetric functions. For the sake of uniformity we adopt the following notation:*

$$E_1 \equiv S_1,\ E_2 \equiv S_2,\ E_3 \equiv S_{123} + S_{234},\ E_4 \equiv S_{1237} + S_{1\cdot 234},$$

$$E_5 \equiv S_{1\cdot 26\cdot 35},\qquad E_6 \equiv S_{14\cdot 35\cdot 36},\qquad E_7 \equiv S_{1234567}.$$

The group G_{168}^7 is the group of the finite geometry of seven points in a plane, transforming three points of this geometry that are on a line into three points on a line, and three points that are not on a line into three points not on a line. Hence, in S_{234} the subscripts run over the triads that are on the seven lines of this geometry and in S_{123} they run over the triads that form

* In these and following identities connecting S- and Σ-polynomials the subscripts themselves are written in place of the letters to which they are attached.

the twenty-one triangles. Thus $S_{234} \equiv x_2x_3x_4 + x_1x_2x_6 + x_1x_3x_5 + x_1x_4x_7 + x_2x_5x_7 + x_3x_6x_7 + x_4x_5x_6$ is the invariant that defines the subgroup G_{168}^7 of $G_{7!}^7$, as well as the finite geometry. Similarly, in S_{1237} the subscripts run over the sets of four points each that form the seven quadrangles of the geometry, etc.

When the letters are affected with exponents, the symmetric polynomials break up still farther into parts that are separately invariant under this group, as follows:

$$\Sigma_1{}^i \equiv S_1{}^i, \quad \Sigma_1{}^i{}_2{}^j \equiv S_1{}^i{}_2{}^j, \quad \Sigma_1{}^i{}_2{}^j{}_3{}^k \equiv S_1{}^i{}_2{}^j{}_3{}^k + S_2{}^i{}_3{}^j{}_4{}^k,$$

$$\Sigma_1{}^i{}_2{}^j{}_3{}^k{}_4{}^l \equiv S_1{}^i \cdot {}_2{}^j{}_3{}^k{}_4{}^l + S_1{}^j \cdot {}_2{}^k{}_3{}^l{}_4{}^i + S_1{}^k \cdot {}_2{}^l{}_3{}^i{}_4{}^j + S_1{}^l \cdot {}_2{}^i{}_3{}^j{}_4{}^k + S_1{}^i{}_2{}^j{}_3{}^k{}_7{}^l,$$

$$\Sigma_1{}^i{}_2{}^j{}_3{}^k{}_4{}^l{}_5{}^m \equiv P(S_1{}^i \cdot {}_2{}^j{}_6{}^k \cdot {}_3{}^l{}_5{}^m),$$

$$\Sigma_1{}^i{}_2{}^j{}_3{}^k{}_4{}^l{}_5{}^m{}_6{}^n \equiv P(S_1{}^i{}_4{}^j \cdot {}_2{}^k{}_5{}^l \cdot {}_3{}^m{}_6{}^n).$$

In the identity of extent five the operator P indicates the sum of all the distinct S-polynomials of extent five obtained by taking the exponents i, j, k, l, and m in groups of two, two, and one, and similarly in the identity of extent six.

Since $\Sigma_1{}^i{}_2{}^j{}_3{}^k{}_4{}^l{}_5{}^m{}_6{}^n{}_7{}^p$ has 5040 terms if all the exponents are different, while any single term has only 168 values under the group, we have

$$\Sigma_1{}^i{}_2{}^j{}_3{}^k{}_4{}^l{}_5{}^m{}_6{}^n{}_7{}^p \equiv P(S_1{}^i{}_2{}^j{}_3{}^k{}_4{}^l{}_5{}^m{}_6{}^n{}_7{}^p).$$

It is clear from their definitions that all the S-polynomials are unaltered by the substitutions of the group G_{168}^7 and that every polynomial that is unaltered by these substitutions is a polynomial in the S-polynomials. We have to determine an irreducible set of S-polynomials in terms of which all S-polynomials can be expressed as polynomial functions.

We may take $x_1, \cdots, x_7$ as the roots of the equation

$$(x - x_1) \cdots (x - x_7) = 0.$$

By means of this equation we may express powers of $x_i \geqq 7$ in terms of the E_i and powers of $x_i \leqq 7$. In the solution of our problem we need therefore consider only those S-polynomials for which $1 \leqq i, j, k, l, m, n, p \leqq 7$.

Our method of procedure is to investigate the S-polynomials in order beginning with these of extents 1 and 2. In the case of

S-polynomials of extent i, $(i=1, \cdots, 7)$, we determine a limited number such that all can be expressed as polynomial functions of this limited number and S-polynomials of extent $>i$. This process terminates when $i=7$, leaving us a limited number of S-polynomials of each extent from which an irreducible set may be selected to form a complete system. The actual work of reduction is very similar to that used in reducing symmetric polynomials. The procedure is illustrated in the following:

$$S_1{}^{i_2}{}^{i_3}{}^{k_7}{}^{l} \equiv S_1{}^{i-1}{}_2{}^{j-1}{}_3{}^{k-1}{}_7{}^{l-1} \cdot S_{1237}$$

minus S-polynomials of extent >4.

By means of this identity we make the reduction of $S_1{}^{i_2}{}^{i_3}{}^{k_7}{}^{l}$ depend on the reduction of $S_1{}^{i_2}{}^{i_3}{}^{k_7}$.

$$S_1{}^{i_2}{}^{i_3}{}^{k_7} \equiv S_1{}^{i-1}{}_2{}^{j-1}{}_3{}^{k-1}{}_7 \cdot S_{123} - S_1{}^{i_2}{}^{i_3}{}^{k-1}{}_7{}^{2} - S_1{}^{i_2}{}^{i-1}{}_3{}^{k_7}{}^{2}$$

$$- S_1{}^{i-1}{}_2{}^{i_3}{}^{k_7}{}^{2} \text{ minus } S\text{-polynomials of extent} > 4$$

$$\equiv S_1{}^{i-1}{}_2{}^{j-1}{}_3{}^{k-1}{}_7 \cdot S_{123} - (S_1{}^{i-1}{}_2{}^{j-1}{}_3{}^{k-2}{}_7 + S_1{}^{i-1}{}_2{}^{j-2}{}_3{}^{k-1}{}_7$$

$+ S_1{}^{i-2}{}_2{}^{j-1}{}_3{}^{k-1}{}_7) \cdot S_{1237}$ minus S-polynomials of extent > 4;

$$S_1{}^{i_2}{}^{i_37} \equiv S_{1237} \cdot \Sigma_1{}^{i-1}{}_2{}^{j-1} \text{ minus } S\text{-polynomials of extent} > 4.$$

The identities exhibited above show that all S-polynomials of the type $S_1{}^{i_2}{}^{i_3}{}^{k_7}{}^{l}$ can be expressed as polynomials in S_{123}, S_{1237}, and E_i, $(i=1, \cdots, 7)$, together with S-polynomials of extent >4. An exactly similar procedure is used for the reduction of all the types of S-polynomials listed above. From the polynomials that are isolated, the following 19, forming an irreducible set, may be conveniently chosen as a complete system for the polynomials that are invariant under the group G_{168}^{7}:

$$E_i, (i = 1, \cdots, 7), \qquad S_{123},$$

$$S_1{}^{j} \cdot_{234}, (j = 1, \cdots, 6), \quad S_1{}^{k} \cdot_{26 \cdot 35}, (k = 2, \cdots, 6).$$

THE UNIVERSITY OF ILLINOIS

CONVERGENCE CRITERIA FOR CONTINUED FRACTIONS*

BY H. S. WALL

1. *Introduction.* The object of this paper is to present two new criteria for the convergence of the continued fraction

$$F(\alpha, z) = \frac{1}{\alpha_1 z} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3 z} + \cdots,$$

in which the numbers α_i are real and different from zero.

Necessary and sufficient conditions for convergence have been discovered for the case that $\alpha_i > 0$ by Stieltjes;† and by Hamburger‡ when $\alpha_{2i+1} > 0$. There seem to be no necessary and sufficient conditions known for the general case, although several sufficient conditions have been found. Van Vleck§ showed that if k is the greatest modulus of the limit points of the numbers $1/(\alpha_i \alpha_{i+1})$, then $F(\alpha, z)$ converges, except for isolated points, within the circle $|z| = 1/(4k)$. Inasmuch as k may be infinite while at the same time $\alpha_i > 0$, $\sum \alpha_i$ diverges, it follows from the work of Stieltjes that $F(\alpha, z)$ may converge to an analytic limit even when the circular region $|z| = 1/(4k)$ vanishes. The theorems which I shall give include certain cases of this sort.

2. *Notation.* Let $a_1, a_2, a_3, \cdots; b_1, b_2, b_3, \cdots$, be two infinite sequences of real non-zero numbers connected by the relations

$$(1) \qquad a_{2i} = b_{2i+1}/(\delta^b_{i-1}\delta^b_i), \qquad a_{2i+1} = b_{2i+2}[\delta^b_i]^2,$$

where

$$\delta^b_i = b_1 + b_3 + \cdots + b_{2i+1}.$$

It is easily seen that if we set

$$g^a_i = a_2 + a_4 + \cdots + a_{2i}, \quad (g^a_0 = 0),$$

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* Presented to the Society, April 3, 1931.

† Annales de la Faculté des Sciences de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47.

‡ Mathematische Annalen, vol. 81, pp. 234–319; vol. 82, pp. 120–187.

§ Transactions of this Society, vol. 2, pp. 476–483.

then

$$\delta_i^b = b_1/(1 - b_1 g_i^a), \tag{2}$$

$$(3)\ b_{2i+1} = b_1^2 a_{2i}/\left[(1 - b_1 g_i^a)(1 - b_1 g_{i-1}^a)\right], b_{2i+2} = a_{2i+1}(1 - b_1 g_i^a)^2/b_1^2.$$

If we are given a set of non-zero a_i, it is clearly possible to find a set of non-zero b_i such that (1) holds. The formulas (3) effect a transformation,

$$b = [a], \tag{4}$$

of the a_i into the b_i. By means of (4) a continued fraction $F(a, z)$ is transformed into another continued fraction $F(b, z)$.

Denote by P_n^a/Q_n^a the nth convergent of $F(\alpha, z)$. Then certain formulas* which I gave in a recent article may be used to connect the polynomials P_n^a, Q_n^a with the P_n^b, Q_n^b. They run as follows:

$$(5)\quad \begin{cases} Q_{2n}^a = Q_{2n+1}^b/(z\delta_n^b), \\ P_{2n}^a = b_1^{-1} Q_{2n}^a - P_{2n+1}^b/\delta_n^b, \\ Q_{2n-1}^a = \delta_n^b Q_{2n}^b - Q_{2n+1}^b/z, \\ P_{2n-1}^a = b_1^{-1} Q_{2n-1}^a - z\left[\delta_n^b P_{2n}^b - P_{2n+1}^b/z\right]. \end{cases}$$

3. *Convergence of* $F(a, z)$. If a continued fraction $F(a, z)$ is transformed by (4) into a continued fraction $F(b, z)$, the equations (5) serve to connect the convergents of $F(a, z)$ with those of $F(b, z)$.

THEOREM 1. *Let* $F(b, z)$ *be the continued fraction obtained from* $F(a, z)$ *by means of the transformation* $b = [a]$, *and suppose that* $\sum |b_i|$ *converges. Then there is a value of* z *for which* $F(a, z)$ *converges if and only if*

$$\lim_{n\to\infty} |g_n^a| = \infty\ . \tag{6}$$

If $F(a, z)$ *converges for a single value of* z, *then there exist two entire functions* $p(z)$ *and* $q(z)$ *such that*

$$\lim_n z P_{2n-1}^a = -\lim_n z\delta_n^b P_{2n}^a = p(z),$$

$$\lim_n z Q_{2n-1}^a = -\lim_n z\delta_n^b Q_{2n}^a = q(z),$$

* Transactions of this Society, vol. 33 (1931), Theorem 1, p. 514.

uniformly over every finite closed region, and hence $F(a, z)$ converges over the entire plane except at isolated points, and its limit is $p(z)/q(z)$.

In fact, by a theorem of von Koch,* it follows from the convergence of $\sum |b_i|$ that the sequences of polynomials

$$P^b_{2n-1},\ Q^b_{2n-1},\ P^b_{2n},\ Q^b_{2n}$$

converge uniformly over every finite closed region to entire limit functions

$$r^b,\ s^b,\ R^b,\ S^b,$$

respectively; and

$$r^b S^b - R^b s^b \equiv +1. \tag{7}$$

Now by (2) and (6),

$$\lim_n \delta^b_n = \delta^b,\quad \delta^b = 0. \tag{8}$$

We find that if (6) does not hold, the only alternative to (8) is that

$$\lim_n \delta^b_n = \delta^b,\quad \delta^b \neq 0, \tag{9}$$

where δ^b is finite.

It now follows immediately from (5) that

$$\lim_n zQ^a_{2n-1} = z\delta^b S^b - s^b = zs^a,$$

$$\lim_n zP^a_{2n-1} = b_1^{-1} zs^a - z[z\delta^b R^b - r^b] = zr^a,$$

$$-\lim_n z\delta^b_n Q^a_{2n} = -s^b = zS^a,$$

$$-\lim_n z\delta^b_n P^a_{2n} = b_1^{-1} zS^a + zr^b = zR^a,$$

uniformly over every finite closed region. When (8) holds we find that $r^a/s^a \equiv R^a/S^a$, and hence $F(a, z)$ converges as stated in the theorem. The entire functions $p(z)$ and $q(z)$ are as follows:

$$p(z) = zr^b - b_1^{-1}s^b,\ q(z) = -s^b.$$

When, on the other hand, (9) holds, we find with the aid of (7)

* Bulletin de la Société Mathématique, vol. 23, pp. 33–40.

that $R^a s^a - r^a S^a = \delta^b$, and therefore if (6) does not hold, $F(a, z)$ diverges for every value of z.*

4. *Another Theorem on Convergence.* Another theorem may be obtained if we transform $F(a, z)$ into $F(b, z)$, and then $F(b, z)$ into $F(c, z)$ by means of the transformations $b = [a]$, $c = [b]$. We find with the aid of (2) and (3) that

$$\delta_i^c = c_1/(1 - c_1 g_i^b), \tag{10}$$

$$\begin{cases} c_{2i+1} = \dfrac{c_1^2 a_{2i-1}(1 - b_1 g_{i-1}^a)^2}{b_1^2(1 - c_1 g_i^b)(1 - c_1 g_{i-1}^b)}, \\ c_{2i+2} = \dfrac{b_1^2 a_{2i}(1 - c_1 g_i^b)^2}{c_1^2(1 - b_1 g_i^a)(1 - b_1 g_{i-1}^a)}. \end{cases} \tag{11}$$

Also by (5),

$$\begin{aligned} Q_{2n}^a &= (\delta_{n+1}^c Q_{2n+2}^c - Q_{2n+3}^c/z)/(z\delta_n^b), \\ P_{2n}^a &= (b_1^{-1} - c_1^{-1} z) Q_{2n}^a + z(\delta_{n+1}^c P_{2n+2}^c - P_{2n+3}^c/z)/\delta_n^b, \\ Q_{2n+1}^a &= \delta_n^b Q_{2n+1}^c/(z\delta_n^c) - \delta_{n+1}^c Q_{2n+2}^c/z + Q_{2n+3}^c/z^2, \\ P_{2n-1}^a &= (b_1^{-1} - c_1^{-1} z) Q_{2n-1}^a + z^2(\delta_n^b P_{2n+1}^c/(z\delta_n^c) - \delta_{n+1}^c P_{2n+2}^c/z + P_{2n+3}^c/z^2). \end{aligned} \tag{12}$$

THEOREM 2. *Let $F(a, z)$, $F(b, z)$, $F(c, z)$ be connected by the relations*

$$b = [a], \quad c = [b],$$

and suppose $\sum |c_i|$ converges. Then $F(a, z)$ converges over the entire plane except at isolated points in the following cases:

(i) $\sum c_{2i+1} = 0$, (ii) $\sum c_{2i+1} \neq 0$, $\sum b_{2i+1} = 0$.

In every other case $F(a, z)$ diverges for all z. In fact, by (12),

$$\frac{P_{2n}^a}{Q_{2n}^a} = b_1^{-1} - c_1^{-1} z + z^2 \left[\frac{z\delta_{n+1}^c P_{2n+2}^c - P_{2n+3}^c}{z\delta_{n+1}^c Q_{2n+2}^c - Q_{2+n3}^c} \right].$$

Now if (i) holds, the numerator and denominator of the fraction on the right converge uniformly over every finite closed region to entire limit functions

$$-r^c, \quad -s^c,$$

* Except possibly at $z=0$. But one may verify directly that, at $z=0$, r^a/s^a has a pole, while R^a/S^a is regular at $z=0$.

respectively. It follows that

$$\lim_n \frac{P^a_{2n}}{Q^a_{2n}} = b_1^{-1} - c_1^{-1}z + z^2\frac{r^c}{s^c},$$

everywhere except for isolated values of z. Again by (12)

$$\frac{P^a_{2n-1}}{Q^a_{2n-1}} = b_1^{-1} - c_1^{-1}z + z^2\left[\frac{zP^c_{2n+1} - z\delta^c_{n+1}(\delta^c_n/\delta^b_n)P^c_{2n+2} + (\delta^c_n/\delta^b_n)P^c_{2n+3}}{zQ^c_{2n+1} - z\delta^c_{n+1}(\delta^c_{2n}/\delta^b_n)Q^c_{n+2} + (\delta^c_n/\delta^b_n)Q^c_{2n+3}}\right].$$

Denote by $f_n(z)$ the fraction on the right, and suppose that z is not a root of s^c. We find that the sequence

$$f_1(z), f_2(z), f_3(z), \cdots$$

is compact. Indeed every infinite subsequence contains a subsequence with the limit r^c/s^c, so that the sequence also converges to the same limit. Hence

$$\lim_n \frac{P^a_{2n-1}}{Q^a_{2n-1}} = b_1^{-1} - c_1^{-1}z + z^2\frac{r^c}{s^c},$$

and therefore $F(a, z)$ converges, except at isolated points.

If (ii) holds, then we find that $\sum |b_i|$ converges, and hence by Theorem 1, $F(a, z)$ converges if and only if $\sum b_{2i+1} = 0$.

5. *Example.* Let

$$\alpha_n = \begin{cases} \sigma^i, & \text{if } n = 2i, \\ \rho^i, & \text{if } n = 2i - 1, \end{cases}$$

where ρ, σ are real and not zero. Then we find that $F(\alpha, z)$ is a meromorphic function by Theorem 1 when

$$|\rho| < 1, \ |1/\sigma| < 1, \ |\rho\sigma^2| < 1;$$

and when

$$|\sigma| < 1, \ |1/\rho| < 1, \ |\rho^2\sigma| < 1,$$

by Theorem 2. Here $\lim_n |1/(\alpha_n\alpha_{n+1})| = \infty$, and the continued fraction does not, in general, satisfy any of the criteria mentioned at the beginning of this article.

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LIMITS OF APPROXIMATE SOLUTIONS OF A TORSION PROBLEM*

BY D. L. HOLL AND E. W. ANDERSON

1. *Introduction.* In the application of Ritz's† method to the solution of a boundary value problem by the direct process of the calculus of variations, one has the difficulty of not knowing what degree of approximation it provides in a given case, except to say that it gives an upper limit to the solution. Analogous to the Ritz method in which the form of the solution (containing parameters determinable by a minimizing process) is approximated by a complete set of independent functions satisfying the boundary conditions but not the differential equation, Trefftz‡ has presented a method adaptable to the Dirichlet problem, and this provides a lower bound. In this process the approximating functions are chosen to satisfy only the differential equation. By application of both of these approximating methods upper and lower bounds of the true minimum are provided.

In this paper these methods are applied to the torsion problem of a right prism whose cross section is symmetrical and is bounded by arcs of two confocal parabolas.

2. *The Torsion Problem.* The torsion problem is the problem§ of determining a function $\phi(x, y)$ such that on the boundary Γ of the region R, it attains the value

$$\phi = \frac{\tau}{2}(x^2 + y^2) + \text{a constant}, \tag{1}$$

and

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0 \tag{2}$$

* Presented to the Society, December 30, 1929. Abstract No. 36–3–145.

† W. Ritz, Journal für Mathematik, vol. 135 (1909), pp. 1–61; Collected Works, 1911, pp. 192–250.

‡ E. Trefftz, *Ein Gegenstück zum Ritzschen Verfahren*, Proceedings of the 2d International Congress for Applied Mechanics, Zurich, 1926, p. 131.

§ A. E. H. Love, *The Mathematical Theory of Elasticity*, 4th edition, 1927, p. 312; Handbuch der Physik, vol. 6, Springer, 1928, p. 143.

holds in R. An equivalent formulation is

$$\psi = \phi - \frac{\tau}{2}(x^2 + y^2) + (\text{a constant}) = 0, \tag{3}$$

$$\nabla^2\psi + 2\tau = 0. \tag{4}$$

The elastic energy per unit length and the twisting couple are, respectively,

$$U = D[\psi] = \frac{\mu}{2}\iint(\psi_x^2 + \psi_y^2)dxdy, \tag{5}$$

$$Q = 2\mu\iint\psi dxdy. \tag{6}$$

For the region R, it is convenient to introduce orthogonal curvilinear coordinates by the transformation

$$2(x + iy) = 2Z = W^2 = (\alpha + i\beta)^2, \tag{7}$$

where constant values of α and β are the boundaries of R. Equations (2) and (5) remain invariant, and the right member of (1) becomes $(\tau/8)(\alpha^2+\beta^2)^2+$a constant.

3. *The Ritz Method.* For the approximate solution, one may assume

$$\psi_n(\alpha, \beta) = \sum_1^n c_i\xi_i(\alpha, \beta) \tag{8}$$

such that each ξ_i vanishes on the boundary, and the c_i are so determined that

$$I[\psi] = \frac{\mu}{2}\iint\left[\psi_\alpha^2 + \psi_\beta^2 - 4\tau\frac{\psi}{h^2}\right]d\alpha d\beta, \quad \left(h = \left|\frac{dW}{dZ}\right|\right), \tag{9}$$

is a minimum. The integral is evaluated over the transformed region R'. Transforming (4) by (7) yields the Euler-Lagrange condition of (9). $I[\psi]$ may also be considered as the equivalent of minimizing $D[\psi]$ subject to a constant torque Q.

It is known that $\psi(\alpha, \beta)$ is such a function that ξ_i may be chosen as a complete set of polynomials which converge uniformly to ψ, such that

$$\lim_{n\to\infty} \psi_n = \psi; \quad \lim_{n\to\infty} I[\psi_n] = I[\psi]; \quad I[\psi_n] \geqq I[\psi]. \tag{10}$$

The convergence of this process has been shown by Trefftz.*

Since R is symmetric, with boundaries $\alpha^2 = \beta^2 = \kappa^2$, take

$$\psi_n = (\alpha^4 - \kappa^4)(\beta^4 - \kappa^4) \sum_0^n c_i \alpha^{2i} \beta^{2i}, \qquad (i = 0, 1, 2, 3, \cdots). \tag{11}$$

Then the equation yields, as a first approximation,

$$\frac{\partial I[\psi_0]}{\partial c_0} = 0$$

$$c_0 = \frac{3\tau}{16\kappa^4}, \quad I_{\min}[\psi_0] = \frac{-4\mu\tau^2\kappa^8}{35}, \quad Q_0 = \frac{8}{35}\mu\tau\kappa^8. \tag{12}$$

Using $i = 0$ and 1 in (11), the second approximation yields

$$c_0 = \frac{209\tau}{1184\kappa^4}, \quad c_1 = \frac{143\tau}{1184\kappa^8}, \tag{13}$$

$$I_{\min}[\psi_1] = -0.11578\mu\tau^2\kappa^8; \quad Q_1 = 0.23155\mu\tau\kappa^8.$$

4. *The Trefftz Method.* For the problem defined by (1) and (2) one may assume approximations of the form

$$\phi_n = \lambda_0 + \sum_{i=1}^{n} \lambda_i p_i(x, y), \tag{14}$$

such that p_i satisfy (2) and the λ factors are determined by

$$\int (\phi_n - \bar{\phi}) ds = 0, \tag{15}$$

$$\int (\phi_n - \bar{\phi}) \frac{\partial p_i}{\partial n} ds = 0, \quad (i = 1, 2, 3, \cdots, n).$$

The normal derivative is along the outward normal and $\bar{\phi}$ is the initial condition (1). Since ϕ is a regular potential function, p_i may be chosen as a complete set of plane harmonic polynomials† such that

* E. Trefftz, *Konvergenz und Fehlerschätzung beim Ritzschen Verfahren*, Mathematische Annalen, vol. 100 (1928), p. 503.

† E. Trefftz, loc. cit.

$$\lim_{n\to\infty} \phi_n = \phi, \quad \lim_{n\to\infty} D[\phi_n] = D[\phi], \quad D[\phi_n] \leqq D[\phi]. \tag{16}$$

The defining relation (3) shows that

$$D[\phi] = I[\psi] + \mu\tau^2 I_0/2,$$

where the last term is a constant and I_0 is the moment of inertia of the section. Hence by this method

$$I[\psi_n] \leqq I[\psi].$$

As a first approximation (14) may be chosen of the form

$$\phi_1 = \lambda_0 + \lambda_1(\alpha^4 - 6\alpha^2\beta^2 + \beta^4); \tag{17}$$

then (15), (9) and (6) yield respectively

$$\lambda_0 = \frac{8}{45}\tau\kappa^4, \quad \lambda_1 = -\frac{5\tau}{72}, \tag{18}$$

$$I_{\max}[\psi_1] = \frac{-16}{135}\mu\tau^2\kappa^8, \quad Q_1 = \frac{32}{135}\mu\tau\kappa^8.$$

Adding $\lambda_2(\alpha^8 - 28\alpha^6\beta^2 + 70\alpha^4\beta^4 - 28\alpha^2\beta^6 + \beta^8)$ to (17) gives the second approximation,

$$\lambda_0 = 0.17856\tau\kappa^4, \quad \lambda_1 = -0.05829\tau, \quad \lambda_2 = 0.00501\tau/\kappa^4; \tag{19}$$

$$I_{\max}[\psi_2] = -0.11690\mu\tau^2\kappa^8, \quad \phi_2 = 0.23380\mu\tau\kappa^8.$$

5. *Results.* The true solution* gives the following results

$$I[\psi] = -0.11589\mu\tau^2\kappa^8, \quad Q = 0.23179\mu\tau\kappa^8.$$

From (12), (13), (18), and (19), the approximate results are

$$I[\psi_0] = -0.11428\mu\tau^2\kappa^8, \quad Q_0 = 0.22857\mu\tau\kappa^8; \tag{12'}$$

$$I[\psi_1] = -0.11578\mu\tau^2\kappa^8, \quad Q_1 = 0.23155\mu\tau\kappa^8; \tag{13'}$$

$$I[\psi_1] = -0.11854\mu\tau^2\kappa^8, \quad Q_1 = 0.23707\mu\tau\kappa^8; \tag{18'}$$

$$I[\psi_2] = -0.11690\mu\tau^2\kappa^8, \quad Q_2 = 0.23380\mu\tau\kappa^8. \tag{19'}$$

From the results it is evident that these methods provide upper and lower bounds to the solution.

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* Anderson and Holl, *A torsion problem in curvilinear coordinates*, Iowa State College Journal of Science, vol. 3 (1929), No. 3, p. 231.

6. *Recent Work.** K. Friedrichs† has shown how to adjoin a maximum problem to a given minimum problem such that its maximum value equals the minimum value of the original problem. It effects the same purpose as the method of Trefftz (furnishing a lower bound) and has wider applications. Associated with the minimizing of $D[\phi]$ subject to the condition (1), is the problem of maximizing

$$-\frac{\mu}{2}\iint (V_x^2 + V_y^2)dxdy + \mu\int \frac{\partial V}{\partial s}\bar{\phi}ds,$$

where V satisfies (2). This coincides with the Trefftz process. Results equivalent to (18) were obtained for this torsion problem.

Dr. N. M. Bashu‡ has employed the Friedrich principle and set up a maximum problem equivalent to $I[\psi]$ subject to $\psi = 0$ on the boundary, and applied it to the deflection of a square membrane and to the torsion of a square prism.

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* Added to the paper since presented to the Society.

† K. Friedrichs, *Ein Verfahren der Variationsrechnung*, · · · , Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, 1929, pp. 13–20; Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. I, 2d edition, 1931, p. 199.

‡ N. M. Bashu, Philosophical Magazine, (7), vol. 10, No. 66, Nov., 1930, p. 886.

THE LATENT ROOTS OF A MATRIX OF SPECIAL TYPE

BY JOHN WILLIAMSON

1. *Introduction.* In this paper the latent roots of a matrix, each of whose elements are suitably restricted rational functions of a square matrix A, are discussed. In particular a theorem is proved about the latent roots of such a matrix, which reduces in the simplest case to the familiar theorem on the value of the latent roots of a rational function of a matrix A. The third section illustrates the fact that the values of circulant determinants, and of determinants composed of blocks which are themselves circulants, can be easily deduced from their matrix properties.

2. *Latent Roots.* Let A be a square matrix of n rows and n columns and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the latent roots of A, where the λ_i need not be distinct. Then by a well known theorem on matrices* there exists a non-singular matrix X, which transforms A into a matrix whose elements in the leading diagonal are the latent roots of A, while all the elements to the left of the leading diagonal are zero. This may be expressed by the matrix equation

$$XAX^{-1} = \begin{pmatrix} \lambda_1 & r_{12} \cdots r_{1n} \\ 0 & \lambda_2 \cdots r_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \\ 0 & 0 \cdots \lambda_n \end{pmatrix} = \Lambda. \tag{1}$$

It follows from (1) that, if $f(A)$ is a rational function of the matrix A, of which the denominator is non-singular, the matrix X transforms the matrix $f(A)$ into a matrix of the same type as Λ, where the elements in the leading diagonal are $f(\lambda_1)$, $f(\lambda_2), \cdots, f(\lambda_n)$. Or

$$Xf(A)X^{-1} = f(\Lambda), \tag{2}$$

* A. Wintner, *Spektraltheorie der unendlichen Matrizen*, pp. 20–22.

where the elements to the left of the leading diagonal in $f(\Lambda)$ are all zero. If now

$$B = (A_{ij}), \qquad (i, j = 1, 2, \cdots, m), \tag{3}$$

denote an m-rowed square matrix all of whose elements are n-rowed square matrices A_{ij}, B is a square matrix of order mn in the elements of the matrices A_{ij}. If

$$Y = (X_{ij}), \qquad (i, j = 1, 2, \cdots, m),$$

is a matrix of the same type as B, where $X_{ij}=0$, if $i \neq j$, and $X_{ii}=X$, then by simple multiplication

$$YBY^{-1} = (XA_{ij}X^{-1}). \tag{4}$$

In particular, if $A_{ij}=f_{ij}(A)$ be rational functions, whose denominators are non-singular, of the original matrix A, then

$$C = (f_{ij}(A)) = M[A] \tag{5}$$

is an nm-rowed square matrix of the same type as B. It follows from equations (2), (4), and (5) that

$$YCY^{-1} = (Xf_{ij}(A)X^{-1}) = (f_{ij}(\Lambda)) = D. \tag{6}$$

Accordingly the latent roots of C are the same as the latent roots of D. But the latent roots of D are the roots of the equation $|D-\lambda E| = 0$, obtained by equating the determinant of the matrix $D-\lambda E$ to zero. If Δ denote the determinant of $D-\lambda E$ and if, for brevity, the element in the ith row and the jth column of Δ is indicated by (i, j), it follows from (2) and (6) that

$$\begin{aligned} &(sn + a, rn + i) = 0, \text{ if } a > i, \qquad (r, s = 0, 1, \cdots, m - 1), \\ &(sn + i, rn + i) = f_{s+1,r+1}(\lambda_i) - \delta_s{}^r\lambda, \qquad (i = 1, 2, \cdots, n). \end{aligned} \tag{7}$$

From Δ a new determinant Δ' is formed by taking for the rows and columns of Δ' in succession the

$$\begin{aligned} &1\text{st}, \quad (n + 1)\text{th}, \ (2n + 1)\text{th}, \cdots, (n(m - 1) + 1)\text{th}, \\ &2\text{d}, \quad (n + 2)\text{th}, \ (2n + 2)\text{th}, \cdots, (n(m - 1) + 2)\text{th}, \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &n\text{th}, \qquad 2n\text{th}, \qquad 3n\text{th}, \cdots, \qquad mn\text{th}, \end{aligned}$$

rows and columns of Δ. Thus, by (7), Δ' is a determinant in which all the elements in the first m columns vanish except

those in the first m rows; all the elements in the second m columns vanish except those in the first $2m$ rows, and in general all the elements in the ith set of m columns vanish except those in the first im rows. Therefore Δ' is the product of the n determinants formed from the first m rows and the first m columns, from the second m rows and the second m columns, etc. But by (7) the determinant formed from the kth set of m columns, and the kth set of m rows of Δ' is

$$\begin{vmatrix} f_{11}(\lambda_k) - \lambda & f_{12}(\lambda_k) & \cdots & f_{1m}(\lambda_k) \\ f_{21}(\lambda_k) & f_{22}(\lambda_k) - \lambda & \cdots & f_{2m}(\lambda_k) \\ \cdot & \cdot & \cdot & \cdot \\ f_{m1}(\lambda_k) & f_{m2}(\lambda_k) & \cdots & f_{mm}(\lambda_k) - \lambda \end{vmatrix},$$

or more shortly $|f_{rs}(\lambda_k) - \delta_r{}^s\lambda|$, $(r, s = 1, 2, \cdots, m)$. Hence with the notation of (5)

$$\Delta' = \prod_{k=1}^{n} |M[\lambda_k] - \lambda E| . \tag{8}$$

But, as Δ differs from Δ' by at most a sign, $\Delta' = 0$ is equivalent to $\Delta = 0$ and accordingly the latent roots of D and therefore the latent roots of C are the nm roots of the n equations obtained by equating the n factors $|M[\lambda_k] - \lambda E|$ of (8) in turn to zero. Since the roots of the equation $|M[\lambda_k] - \lambda E| = 0$ are the latent roots of the matrix $M[\lambda_k]$, the following theorem is true.

THEOREM 1. *If A be an n-rowed square matrix, whose latent roots are $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $C = (f_{rs}(A))$ be a square matrix whose m^2 elements are themselves n-rowed square matrices, which are rational functions, with non-singular denominators, of the original matrix A, then the latent roots of the matrix C are the nm latent roots of the n m-rowed square matrices $(f_{rs}(\lambda_k))$, $(k = 1, 2, \cdots, n)$.*

COROLLARY 1. *If $m = 1$, this theorem reduces to the well known theorem that the latent roots of $f_{11}(A)$ are $f_{11}(\lambda_k)$, for the latent root of the one-rowed square matrix $f_{11}(\lambda_k)$ is obviously $f_{11}(\lambda_k)$.*

COROLLARY 2. *If $n = 1$, the square matrix A may be taken to be the number 1 and the latent root of 1, considered as a matrix, as 1 itself, so that in this case the theorem simply reduces to the definition of the latent roots of a matrix.*

If $F(M[\lambda]) = N[\lambda] = (g_{rs}(\lambda))$ is a matrix polynomial in the matrix $M[\lambda]$, it is easily shown that $F(C) = N[A] = (g_{rs}(A))$. Hence by Theorem 1 the latent roots of $F(C)$ are the nm latent roots of the n m-rowed matrices $F(M[\lambda_k])$, $(k = 1, 2, \cdots, n)$. In fact if $F(C)$ is a rational function of C of which the denominator $H(C)$ is non-singular, the latent roots of $F(C)$ are the nm latent roots of the n matrices $F(M[\lambda_k])$. But, the determinant of $H(C)$ is the product of the latent roots of $H(C)$ and therefore is equal to the product of the determinants of the n matrices $H(M[\lambda_k])$. Hence $H(C)$ is non-singular if the n matrices $H(M[\lambda_k])$ are non-singular. Accordingly this proves the following theorem.

THEOREM 2. *If A and C are the matrices of Theorem* 1 *and if $F(C)$ is a rational function of C with a non-singular denominator, the latent roots of $F(C)$ are the nm latent roots of the n matrices $F(M[\lambda_k])$, $(k = 1, 2, \cdots, n)$. Moreover, the denominator of $F(C)$ is non-singular if and only if the denominators of all the matrices $F(M[\lambda_k])$ are non-singular.*

Theorem 1, though perhaps presented in its most interesting form, is a particular case of a more general theorem. For the proof of Theorem 1 it is not essential that the functions $f_{ij}(A)$ be functions of the same matrix A. In fact for $f_{ij}(A)$ in Theorem 1 we may write $f_{ij}(A_{ij})$ without essentially altering the proof, provided that the same matrix X transforms all the matrices A_{ij} into matrices of type Λ. Thus we have the following theorem.

THEOREM 3. *If A_{ij} is an n-rowed square matrix, whose latent roots are $\lambda_{ij}^{(1)}, \lambda_{ij}^{(2)}, \cdots, \lambda_{ij}^{(n)}$, and if for all values of $i, j = 1, 2, \cdots, m$ the same non-singular matrix X transforms the matrix A_{ij} into a matrix of type Λ with $\lambda_{ij}^{(1)}, \lambda_{ij}^{(2)}, \cdots, \lambda_{ij}^{(n)}$ as its leading diagonal, then the latent roots of the matrix*

$$D = (f_{ij}(A_{ij})), \qquad (i, j = 1, 2, \cdots, m),$$

are the nm latent roots of the n m-rowed square matrices

$$(f_{ij}(\lambda_{ij}^{(k)})), \qquad (k = 1, 2, \cdots, n).$$

3. *Applications.* Let e_{ij}, $(i, j = 1, 2, \cdots, n)$ denote the matrix whose element in the ith row and the jth column is 1, while all other elements are zero. Further let e_{ks} denote e_{ij} if $k > n$, $s > n$, but $k \equiv i$ and $s \equiv j$ mod n. Thus the matrices E_r defined by

$$E_r = \sum_{i=1}^{n} e_{i,i+r} = e_{\alpha,\alpha+r}, \tag{9}$$

where the α's are summation suffixes, are non-singular matrices. Since the matrices e_{ij} obey the rule of multiplication, we have

$$e_{ij}e_{km} = \delta_j{}^k e_{im}, \tag{10}$$

$$E_rE_s = e_{\alpha,\alpha+r}e_{\beta,\beta+s} = e_{\alpha,\alpha+r+s} = E_{r+s}. \tag{11}$$

Hence by repeated applications of (11) with $s=1$ and $r=1, 2, \cdots, r-1$,

$$E_1{}^r = E_r, \quad E_1{}^n = E_0 = E, \tag{12}$$

the unit matrix.

If now

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1},$$

by (12)

$$f(E_1) = a_0E_0 + a_1E_1 + a_2E_2 + \cdots + a_{n-1}E_{n-1}.$$

But by (9)

$$f(E_1) = \begin{pmatrix} a_0 & a_1 & a_2 \cdots a_{n-1} \\ a_{n-1} & a_0 & a_1 \cdots a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 \cdots a_{n-3} \\ \cdots & \cdots & \cdots \\ a_1 & a_2 & a_3 \cdots a_0 \end{pmatrix},$$

that is, $f(E_1)$ is a matrix whose determinant is a circulant and may be for brevity called a circulant matrix. If $f_{ij}(x) = \sum_{k=0}^{n-1} a_k^{ij}x^k$ and $A = E_1$, the matrix C defined by (5) is a matrix whose component matrices $f_{ij}(E_1)$ are all circulant matrices. By Theorem 1 the latent roots of C are the nm latent roots of the matrices $M[\lambda_k]$, where the λ_k are the latent roots of E_1. But the latent roots of E_1 are the nth roots of unity $\omega_1, \omega_2, \cdots, \omega_n$. Hence the latent roots of C are the nm latent roots of the n matrices $M[\omega_k]$, $k=1, 2, \cdots, n$. In particular the determinant of C is the product of the latent roots of C so that

$$|C| = \prod_{k=1}^{n} |M[\omega_k]| = \prod_{k=1}^{n} |f_{ij}(\omega_k)|. \tag{13}$$

If $m=1$ in (13), this result reduces to the well known theorem on the value of a circulant,* while if $m>1$ formula (13) is the statement of a result proved by Zariski on the value of a determinant composed of circulants.†

If, with the previous interpretation of the notation e_{ij}, we define

$$E_1 = \sum_{i=1}^{n} x_i e_{i,i+1},$$

then

$$E_1^r = \sum_{i=1}^{n} x_i x_{i+1} \cdots x_{i+r-1} e_{i,i+r},$$

where $x_i = x_j$, if $i \equiv j \bmod n$, and Theorem 1 gives the value of another type of determinant, which, for $n=4$, is the determinant of the matrix

$$(14)\ f(E_1) = \begin{pmatrix} a_0x_1x_2x_3x_4 & a_1x_1 & a_2x_1x_2 & a_3x_1x_2x_3 \\ a_3x_2x_3x_4 & a_0x_2x_3x_4x_1 & a_1x_2 & a_2x_2x_3 \\ a_2x_3x_4 & a_3x_3x_4x_1 & a_0x_3x_4x_1x_2 & a_1x_3 \\ a_1x_4 & a_2x_4x_1 & a_3x_4x_1x_2 & a_0x_4x_1x_2x_3 \end{pmatrix}.$$

Since the latent roots of E_1 are now ρ_k, where the ρ_k are the nth roots of the product $x_1x_2, \cdots, x_n$, the value of the determinant of such a matrix is $\prod_{k=1}^{n} f'(\rho_k)$, where $f'(x)$ is the polynomial $f(x)$ with its constant term replaced by a_0x^n. If $E_1=A$, but $m>1$, Theorem 1 gives the value of a determinant composed of m^2 blocks of the type (14), where the a_i may vary from block to block, but the x_i must remain the same.

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* Thomas Muir, *Theory of Determinants*, vols. 1, 2, 3. Under the heading of circulants many determinants which are special cases of Zariski's are discussed.

† O. Zariski, *On the linear connection index of the algebraic surfaces* $z^n=f(x, y)$, Proceedings of the National Academy of Sciences, vol. 15 (1929), No. 6, p. 497.

NOTE ON A THEOREM OF BÔCHER AND KOEBE*

BY J. J. GERGEN

1. *Introduction.* In this paper a generalization of the following theorem, discovered independently by Bôcher† and Koebe,‡ is established.

THEOREM 1. *If $u(x, y)$ is continuous with its first partial derivatives in a plane region R, and if, for every circle C contained in R,*

$$\int_C \frac{\partial u}{\partial n} ds = 0,$$

where n is the exterior normal to C, then u is harmonic in R.

The generalization obtained is embodied in Theorem 2.

THEOREM 2. *If $v(x, y)$ is harmonic and positive in R, if $u(x, y)$ is continuous with its first partial derivatives in R, and if*

$$\int_C v \frac{\partial u}{\partial n} ds = \int_C u \frac{\partial v}{\partial n} ds \tag{1}$$

for every circle C contained in R, then u is harmonic in R.

Taking v as the constant one in Theorem 2, Theorem 1 is obtained.

Like Theorem 1,§ Theorem 2 has an analog in space, but,

* Presented to the Society, April 3, 1931.

† Bôcher, M., *On harmonic functions in two dimensions*, Proceedings of the American Academy of Arts and Sciences, vol. 41 (1906), pp. 577–583.

‡ Koebe, P., *Herleitung der partiellen Differentialgleichung der Potentialfunktion aus der Intergraleigenschaft*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, vol. 5 (1906), pp. 39–42.

§ Koebe, loc. cit. For generalizations of Bôcher's and Koebe's Theorem of another type, see G. C. Evans, *Fundamental points of potential theory*, Rice Institute Pamphlets, vol. 7 (1920), pp. 252–329, especially p. 286, and *Note on a theorem of Bôcher*, American Journal of Mathematics, vol. 50 (1928), pp. 123–126; and G. E. Raynor, *On the integro-differential equation of the Bôcher type in three space*, this Bulletin, vol. 52 (1926), pp. 654–658. Evans, using the notion of the potential function of a gradient vector, shows that the conclusion of Theorem 1 holds with much lighter hypotheses both on u and the character of the curves C.

since no new essentially different details present themselves in the proof for space, we simply state this analog, and consider in detail only the plane case.

THEOREM 3. *If $v(x, y, z)$ is harmonic and positive in a region R in space, if $u(x, y, z)$ is continuous with its first partial derivatives in R, and if, for every sphere C contained in R,*

$$\iint_C v \frac{\partial u}{\partial n}\, ds = \iint_C u \frac{\partial v}{\partial n}\, ds,$$

where n is the exterior normal to C, then u is harmonic in R.

The proof of Theorem 2 is elementary in character. The idea is to express uv as a sum of integrals and deduce the character of u from the properties of these integrals.

2. *Proof of Theorem* 2. We first observe that it is enough to prove the theorem in the case that R is the interior of a circle C, and the hypotheses hold in the interior R' and on the boundary* of a circle C' concentric with C but of larger radius. The problem, then, is to show that u_{xx} and u_{yy} exist and are continuous in R, and that

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{2}$$

there.

Let $P(x, y)$ be any point in R. Let α' be the radius of C', α the radius of C, and

$$\rho = \tfrac{1}{2}(\alpha' - \alpha).$$

Then, by (1), the hypothesis on v, and a classical formula, we have, for $0 < t \leqq \rho$,

$$\int_{C(P,t)} v \frac{\partial u}{\partial n}\, ds = \int_{C(P,t)} u \frac{\partial v}{\partial n}\, ds = \iint_{\sigma(P,t)} \phi\, d\sigma, \tag{3}$$

where $\sigma(P, t)$ is the interior of the circle $C(P, t)$ of radius t about P, and

$$\phi(x, y) = \nabla u \cdot \nabla v = u_x v_x + u_y v_y.$$

* That is to say, the hypotheses hold in a region containing R' and its boundary.

It is the third integral in (3) that enables us to express uv as a sum of integrals, whose properties lead to the conclusion of the theorem. Writing

$$\sigma = \sigma(P, \rho), \quad S' = R' + C', \quad r = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2},$$

we find that

$$\begin{aligned}
&\pi\rho^2 u(x, y)v(x, y) \\
&\quad = \left\{ \iint_\sigma uv\,d\sigma + \rho^2\left(\frac{1}{2} + \log\frac{1}{\rho}\right)\iint_\sigma \phi\,d\sigma \right. \\
&\quad\quad \left. - \frac{1}{2}\iint_\sigma \phi(\xi, \eta) r^2\,d\xi\,d\eta - \rho^2 \iint_{S'-\sigma} \phi \log r\,d\sigma \right\} \qquad (4) \\
&\quad\quad + \rho^2 \iint_{S'} \phi \log r\,d\sigma \\
&\quad = J'(P) + J''(P), \text{ say.}
\end{aligned}$$

In fact, by (3),

$$\begin{aligned}
\int_0^\rho \tau\,d\tau \int_0^\tau \frac{dt}{t} \int_{C(P,t)} v\frac{\partial u}{\partial n}\,ds &= \int_0^\rho \tau\,d\tau \int_0^\tau \frac{dt}{t} \int_{C(P,t)} u\frac{\partial v}{\partial n}\,ds \\
&= \int_0^\rho \tau\,d\tau \int_0^\tau \frac{dt}{t} \iint_{\sigma(P,t)} \phi\,d\sigma,
\end{aligned}$$

or $K_1(P) = K_2(P) = K_3(P)$, say. Introducing, then, a system of polar coordinates (r, θ) with pole at P, we have

$$\begin{aligned}
K_1(P) &= \int_0^\rho \tau\,d\tau \int_0^{2\pi} d\theta \int_0^\tau v\frac{\partial u}{\partial r}\,dr \\
&= \int_0^\rho \tau\,d\tau \int_0^{2\pi} \left\{ uv - u(x, y)v(x, y) - \int_0^\tau u\frac{\partial v}{\partial r}\,dr \right\} d\theta \\
&= \iint_\sigma uv\,d\sigma - \pi\rho^2 u(x, y)v(x, y) - K_2(P),
\end{aligned}$$

so that

$$\pi\rho^2 u(x, y)v(x, y) = \iint_\sigma uv\,d\sigma - 2K_3(P).$$

But, we see that

$$K_2(P) = \int_0^{2\pi} d\theta \int_0^{\rho} \tau d\tau \int_0^{\tau} \frac{dt}{t} \int_0^{t} r\phi dr$$

$$= \int_0^{2\pi} d\theta \int_0^{\rho} \left\{ \tfrac{1}{2}\rho^2(\log \rho - \tfrac{1}{2}) + \tfrac{1}{4}r^2 - \tfrac{1}{2}\rho^2 \log r \right\} r\phi dr,$$

upon changing the order of integration twice. Hence (4) follows.

Consider, now, the derivatives of J'. We have

$$J'(P) = \int_{y-\rho}^{y+\rho} d\eta \int_{x-\psi}^{x+\psi} w d\xi + \left\{ \int_{-\alpha'}^{y-\rho} d\eta \int_{-\psi'}^{\psi'} + \int_{y-\rho}^{y+\rho} d\eta \int_{-\psi'}^{x-\psi} \right.$$

$$\left. + \int_{y-\rho}^{y+\rho} d\eta \int_{x+\psi}^{\psi'} + \int_{y+\rho}^{\alpha'} d\eta \int_{-\psi'}^{\psi'} \right\} w' d\xi,$$

where

$$w = w(x, y; \xi, \eta) = u(\xi, \eta)v(\xi, \eta) + \phi(\xi, \eta)\left\{\tfrac{1}{2}\rho^2 - \rho^2 \log \rho - \tfrac{1}{2}r^2\right\},$$

$$w' = w'(x, y; \xi, \eta) = -\rho^2\phi(\xi, \eta) \log r,$$

$$\psi = \psi(\eta, y) = \left\{\rho^2 - (\eta - y)^2\right\}^{1/2}, \psi' = \psi'(\eta) = (\alpha'^2 - \eta^2)^{1/2}.$$

We see, then, by using the fact that u, v and ϕ are continuous in S', and the formula for differentiation under the integral sign, that, for P in R, J_x' exists and is given by

$$J_x' = \int_{y-\rho}^{y+\rho} \left\{ w(x, y; x + \psi, \eta) - w(x, y; x - \psi, \eta) \right\} d\eta$$

$$+ \int_{y-\rho}^{y+\rho} \left\{ w'(x, y; x - \psi, \eta) - w'(x, y; x + \psi, \eta) \right\} d\eta$$

$$+ \iint_{\sigma} w_x d\sigma + \iint_{S'-\sigma} w_x' d\sigma$$

$$= \int_{y-\rho}^{y+\rho} \left\{ u(x + \psi, \eta)v(x + \psi, \eta) - u(x - \psi, \eta)v(x - \psi, \eta) \right\} d\eta$$

$$- \iint_{\sigma} (x - \xi)\phi d\sigma - \rho^2 \iint_{S'-\sigma} \phi \frac{(x - \xi)}{r^2} d\sigma;$$

which reduces to

$$(5)\quad J_x' = \int\int_\sigma \{uv_x + vu_x - (x-\xi)\phi\}\,d\sigma - \rho^2 \int\int_{S'-\sigma} \phi \frac{(x-\xi)}{r^2}\,d\sigma,$$

upon writing

$$u(x+\psi,\eta)v(x+\psi,\eta) - u(x-\psi,\eta)v(x-\psi,\eta) = \int_{x-\psi}^{x+\psi} \{uv_x + vu_x\}\,d\xi.$$

From (5) and the continuity of u and v and their first partial derivatives, we deduce immediately that J_{xx}' and, as it is worth while noting for future purposes, J_{xy}' exist and are continuous in R. By analogy, J_{yy}' exists and is continuous in R.

Next, consider J''. The existence of J_x'' and J_y'' can be inferred from (4) and the existence of J_x', J_y' and the first partial derivatives of u and v. We wish to know, further, that J_{xx}'' and J_{yy}'' exist and are continuous in R. To prove this, we first observe that $u, v^{-1}, J_x', J_y', J_x'', J_y''$ satisfy uniform Hölder conditions* in any closed domain S'' bounded by a circle C'' contained in R. The first four of these functions have this property because their first partial derivatives are continuous in R and R is convex† and contains S'', the last two because of a theorem of Dini.‡ We next observe that from this property of $u, v^{-1}, \cdots, J_y''$, it follows that ϕ satisfies a uniform Hölder conditionin S'', for

$$\phi = \{\nabla v \cdot \nabla J' + \nabla v \cdot \nabla J'' - \pi\rho^2 u \nabla v \cdot \nabla v\}/(\pi\rho^2 v),$$

and thus ϕ is equal to a combination of sums and products of functions each of which satisfies a uniform Hölder condition in

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* A function $f(P)$, defined on a set E, satisfies a uniform Hölder condition on E if, P and Q being any two points of E,

$$|f(P) - f(Q)| < A\,|PQ|^\lambda,$$

where A and λ are independent of P and Q, and $\lambda > 0$. Evidently, if $f_1(P)$ and $f_2(P)$ satisfy uniform Hölder conditions on E, $f_1 + f_2$ and $f_1 f_2$ have the same property.

† A region R is convex if each segment, whose end points lie in R, lies in R.

‡ U. Dini, *Sur la méthode des approximations successives pour les équations aux dérivées partielles du deuxième ordre*, Acta Mathematica, vol. 25 (1901), pp. 185–230. The function J'' is, of course, the potential function due to a distribution of continuous density $-\rho^2\phi$ over S'.

S''. The existence and continuity of J_{xx}'' and J_{yy}'' can now readily be deduced. We write

$$J'' = \rho^2 \iint_{S'-S''} \phi \log r d\sigma + \rho^2 \iint_{S''} \phi \log r d\sigma$$
$$= L' + L'', \text{ say}.$$

Now L_{xx}' and L_{yy}' evidently exist and are continuous for P in the interior of S'', while L_{xx}'' and L_{yy}'' exist and are continuous in S'' by a theorem of Hölder.* Thus J_{xx}'' and J_{yy}'' exist and are continuous in the interior of S''. But C'' was arbitrary in R; and hence it follows that J_{xx}'' and J_{yy}'' exist and are continuous in R.

The proof is now almost complete. Since J_{xx}', J_{yy}', J_{xx}'', J_{yy}'' exist and are continuous in R, and since v is positive and harmonic there, u_{xx} and u_{yy} exist and are continuous in R. It remains, then, only to prove that (2) holds. To prove this, we show that a contrary assumption leads to a contradiction. Suppose there is a point P in R at which $|\nabla^2 u| = 2\beta$ is different from zero. Then we can choose t so small that $\sigma(P, t)$ lies in R and $|\nabla^2 u| > \beta$ in $\sigma(P, t)$. Thus, since v exceeds a positive constant ϵ in $\sigma(P, t)$ and $\nabla^2 u$ is continuous there,

$$\left| \iint_{\sigma(P,t)} v \nabla^2 u d\sigma \right| \geqq \epsilon\beta \text{ area } \sigma(P, t) > 0. \tag{6}$$

But

$$\iint_{\sigma(P,t)} v \nabla^2 u d\sigma = \int_{C(P,t)} v \frac{\partial u}{\partial n} ds - \int_{C(P,t)} u \frac{\partial v}{\partial n} ds = 0, \tag{7}$$

by the continuity of $\nabla^2 u$, our hypotheses, and Green's formula. In (6) and (7) we reach the desired contradiction. The proof is now complete.

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* Otto Hölder, *Beiträge zur Potentialtheorie*, Inaugural Dissertation, Stuttgart, 1882, p. 17. Hölder considers the second partial derivatives of a volume distribution. The same type of analysis, however, holds for plane distributions.

ON COMPLEX METHODS OF SUMMABILITY*

BY R. P. AGNEW†

1. *Introduction.* When a method of summability‡ evaluates a complex sequence $\{s_n\}$ to L, it is of interest to know if that method evaluates $\{\mathcal{R}(s_n)\}$ to $\mathcal{R}(L)$, if it evaluates $\{\mathcal{J}(s_n)\}$ to $\mathcal{J}(L)$, and if it evaluates $\{\bar{s}_n\}$ to $\bar{L}$.§ For linear methods of summability,‖ these three questions are easily shown to be equivalent.

To simplify our discussion, we introduce the two following definitions. *A method of summability has property* A *if, corresponding to each sequence* $\{s_n\}$ *which it evaluates, the sequence* $\{\mathcal{R}(s_n)\}$ *is evaluated to the real part of the value of* $\{s_n\}$. *A method of summability has property* B *if, corresponding to each bounded sequence* $\{s_n\}$ *which it evaluates, the sequence* $\{\mathcal{R}(s_n)\}$ *is evaluated to the real part of the value of* $\{s_n\}$.

2. *Failure of Property* B. That a linear regular method may fail to have property B, and hence a fortiori fail to have property A, follows easily from a consideration of the transformation¶

$$(1) \qquad \sigma_n = \tfrac{1}{2}[1 - (-1)^n i]\, s_{n-1} + \tfrac{1}{2}[1 + (-1)^n i]\, s_n$$

which assigns to a given sequence $\{s_n\}$ the value $\lim \sigma_n$ when this limit exists. The bounded sequence $\{x_n\}$ defined by $x_n = 1 + (-1)^n i$ is evaluated to 0 by (1); but $\{\mathcal{R}(x_n)\}$ is evaluated to 1, $\{\mathcal{J}(x_n)\}$ is evaluated to -1, and $\{\bar{x}_n\}$ is evaluated to 2. This

* Presented to the Society, June 13, 1931.

† National Research Fellow.

‡ By a *method of summability*, we mean simply a rule which assigns to each given sequence (or series) of complex numbers either no value or a single value. For example, if we agree to assign to the complex sequence $\{s_n\}$, where $s_n = u_n + iv_n$, u_n and v_n real, the value $3+4i$ if $v_n \neq 0$ for some n and the value 3 if $v_n = 0$ for all n, we have a method of summability.

§ If $w = u + iv$, where u and v are real, we use $\mathcal{R}(w)$, $\mathcal{J}(w)$, and $\bar{w}$ to denote respectively u, iv, and the conjugate $u - iv$ of w.

‖ For definitions of *linearity*, *regularity*, etc., and for necessary and sufficient conditions for regularity, see an expository paper by W. A. Hurwitz, this Bulletin, vol. 28 (1922), pp. 17–36, and the references there given.

¶ Corresponding to a given sequence $s_1, s_2, s_3, \cdots$, we define $s_0 = 0$.

example is of especial interest in that $\{\mathcal{R}(x_n)\}$ not only fails to be summable to the real part of the value of $\{x_n\}$, but is actually summable to a different value.

It will appear later (§8) that a linear regular method may have property B and fail to have property A.

3. *Necessary Condition for Properties* A *and* B. The following two theorems, of which the proofs are immediate, give conditions which are necessary in order that a method may have property A or property B.

THEOREM 1. *If a method having property* A *evaluates a real sequence, the value assigned must be real.*

THEOREM 2. *If a method having property* B *evaluates a real bounded sequence, the value assigned must be real.*

The converses of these theorems do not hold as we shall now show by giving an example of a linear regular transformation which assigns a real value to each real sequence which it evaluates, but which nevertheless fails to have properties A and B. The transformation is

$$\sigma_n = \tfrac{1}{2}(1 - i)s_{n-1} + \tfrac{1}{2}(1 + i)s_n. \tag{2}$$

To show that (2) assigns a real value to each real sequence which it evaluates, let $\{x_n\}$ be a real sequence evaluated by (2) to L_1+iL_2. Then $\lim \frac{1}{2}(x_n+x_{n-1})=L_1$ and $\lim \frac{1}{2}(x_n-x_{n-1})=L_2$, and on adding and subtracting we find $x_n \to L_1+L_2$ and $x_{n-1}\to L_1-L_2$; hence $L_1+L_2=L_1-L_2$ so that $L_2=0$ and the reality of the value is established. To show that (2) does not have properties A and B, we consider the sequence $\{y_n\}$ defined by

$$y_{4p-3} = 1 - i,\ y_{4p-2} = 1 + i,\ y_{4p-1} = -1 + i,\ y_{4p} = -1 - i,$$
$$(p = 1, 2, 3, \cdots).$$

It is easily verified that (2) evaluates $\{y_n\}$ to 0, and that (2) fails to evaluate $\{\mathcal{R}(y_n)\}$; hence, since $\{y_n\}$ is bounded, (2) fails to have properties A and B.

4. *Complex Transformations and Associated Real Transformations.* We now consider methods of summability which involve transformations (G) defined as follows. Let T be a metric set having a limit point t_0 not belonging to T, and let complex

functions $a_k(t)$, $k=1, 2, 3, \cdots$, be defined for all t in T. If a sequence s_n is such that

$$\text{(G)} \qquad \sigma(t) = \sum_{k=1}^{\infty} a_k(t)s_k$$

converges for all t in T, and if

$$\lim_{t \to t_0(T)} \sigma(t) = L,$$

then $\{s_n\}$ is said to be summable by the method or transformation (G), or simply summable (G), to L.

Corresponding to a given (G), we define $b_k(t)$ and $c_k(t)$ to be the real functions determined by the equations

$$a_k(t) = b_k(t) + ic_k(t).$$

The transformation

$$\mathfrak{R}(\text{G}) \qquad \sigma(t) = \sum_{k=1}^{\infty} b_k(t)s_k$$

may be called the *associated real transformation* of (G). Using the well known necessary and sufficient conditions for regularity of (G), we obtain the result: *If* (G) *is regular, then its associated real transformation* $\mathfrak{R}$(G) *is also regular.*

5. *Properties of Transformations.* The following two theorems serve to establish the equivalence of certain problems involving properties of (G).

THEOREM 3. *In order that* (G) *may have property* A, *it is necessary and sufficient that* $\mathfrak{R}$(G) *include* (G).

THEOREM 4. *In order that* (G) *may have property* B, *it is necessary and sufficient that* $\mathfrak{R}$(G) *include* (G) *over the set of all bounded sequences.**

We will now prove Theorem 3; the same method, together with the assumption that all sequences considered are bounded, furnishes a proof of Theorem 4.

To establish sufficiency, let $\mathfrak{R}$(G) include (G) and let (G) evaluate a given sequence $\{s_n\}$ to L; we are to show that (G)

* We say that a method includes a second method over a set S of sequences if each element of S which is summable by the second method is also summable to the same value by the first method.

evaluates $\{\mathcal{R}(s_n)\}$ to $\mathcal{R}(L)$. It then follows from the definition of summability (G) that the series

$$\sum_{k=1}^{\infty} a_k(t)s_k, \quad \sum_{k=1}^{\infty} b_k(t)s_k$$

both converge over T, and that the value of each approaches L as $t \to t_0$; hence $\sum_{k=1}^{\infty} c_k(t)s_k$ converges over T, and its value $\to 0$ as $t \to t_0$. Since $b_k(t)$ and $c_k(t)$ are real, it follows that the series

$$\sum_{k=1}^{\infty} b_k(t)\mathcal{R}(s_k), \quad \sum_{k=1}^{\infty} c_k(t)\mathcal{R}(s_k)$$

both converge over T, and that their values approach $\mathcal{R}(L)$ and 0 respectively as $t \to t_0$. Hence the series

$$\sum_{k=1}^{\infty} a_k(t)\mathcal{R}(s_k) \equiv \sum_{k=1}^{\infty} [b_k(t) + ic_k(t)]\mathcal{R}(s_k)$$

converges over T and its value approaches $\mathcal{R}(L)$ as $t \to t_0$. Thus sufficiency is proved.

To establish necessity, let (G) have property A, and let $\{s_n\}$ be a sequence which (G) evaluates to L; we are to show that $\mathcal{R}(G)$ evaluates $\{s_n\}$ to L. We find that the series

$$\sum_{k=1}^{\infty} a_k(t)s_k, \quad \sum_{k=1}^{\infty} a_k(t)\mathcal{R}(s_k)$$

both converge over T, and their values approach L and $\mathcal{R}(L)$ respectively as $t \to t_0$. Hence $\sum_{k=1}^{\infty} a_k(t)\mathcal{J}(s_k)$ converges over T and approaches $\mathcal{J}(L)$ as $t \to t_0$. Since $\mathcal{R}(s_k)$ is real and $\mathcal{J}(s_k)$ is pure imaginary, it follows that the series

$$\sum_{k=1}^{\infty} b_k(t)\mathcal{R}(s_k), \quad \sum_{k=1}^{\infty} b_k(t)\mathcal{J}(s_k)$$

both converge over T and that their values approach $\mathcal{R}(L)$ and $\mathcal{J}(L)$ respectively as $t \to t_0$. Hence the series

$$\sum_{k=1}^{\infty} b_k(t)s_k \equiv \sum_{k=1}^{\infty} b_k(t)[\mathcal{R}(s_k) + \mathcal{J}(s_k)]$$

converges over T and approaches $\mathcal{R}(L) + \mathcal{J}(L) = L$ as $t \to t_0$. Thus necessity is proved.

6. THEOREM 5. *If* (G) *satisfies the conditions*

$$\sum_{k=1}^{\infty} |c_k(t)| \text{ converges over } T, \tag{3}$$

$$\lim_{t \to t_0(T)} \sum_{k=1}^{\infty} |c_k(t)| = 0, \tag{4}$$

then (G) *and* $\mathcal{R}$(G) *are equivalent over the set of all bounded sequences.**

Let $\{s_n\}$ be any bounded sequence. It follows from (3) and (4) that the series $\sum_{k=1}^{\infty} c_k(t)s_k$ converges over T and that its value approaches 0 as $t \to t_0$. Hence if either of the series

$$\sum_{k=1}^{\infty} a_k(t)s_k, \qquad \sum_{k=1}^{\infty} b_k(t)s_k$$

converges over T, the other must also converge over T and we may write

$$\sum_{k=1}^{\infty} a_k(t)s_k = \sum_{k=1}^{\infty} b_k(t)s_k + i \sum_{k=1}^{\infty} c_k(t)s_k.$$

It follows that if the value of either of the series approaches a limit as $t \to t_0$, the value of the other series must approach the same limit and the theorem is proved.

7. *Sufficient Condition for Property* B. Combining Theorems 4 and 5, we obtain the following result.

THEOREM 6. *If* (G) *satisfies* (3) *and* (4), *then* (G) *has property* B.

Since (G) is linear, the preceding theorem may be amplified to produce the following theorem.

THEOREM 7. *If* (G) *satisfies* (3) *and* (4), *and* $\{s_n\}$ *is a bounded sequence which* (G) *evaluates, say to* L, *then* $\{\mathcal{R}(s_n)\}$, $\{\mathcal{J}(s_n)\}$ *and* $\{\bar{s}_n\}$ *are evaluated by* (G) *to* $\mathcal{R}(L)$, $\mathcal{J}(L)$, *and* $\bar{L}$ *respectively.*

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* The conditions (3) and (4) are of course not sufficient to ensure regularity of (G); hence this theorem and its applications give information concerning non-regular transformations. On the other hand, (3) is satisfied by every regular (G), and (4) is satisfied by every regular (G) which satisfies the important and useful condition $\lim_{t \to t_0(T)} \sum_{k=1}^{\infty} |a_k(t)| = 1$.

8. *Examples.* We will now give an example of a linear regular transformation of the form (G) which has property B and fails to have property A. This example shows that Theorems 2, 4, 5, and 7 do not hold if the word "bounded" is omitted, and that B cannot be replaced by A in Theorem 6. The transformation is

$$(\mathrm{H}) \quad \sigma_n = \frac{1}{2}\left[1 - \frac{(-1)^n i}{\log (n+1)}\right] s_{n-1} + \frac{1}{2}\left[1 + \frac{(-1)^n i}{\log (n+2)}\right] s_n,$$

which assigns to a given sequence $\{s_n\}$ the value lim σ_n when this limit exists. That (H) is of the form (G) is seen by taking (T) to be the set of positive integers, t_0 to be the symbolic limit point $+\infty$, and by writing σ_n for $\sigma(n)$. Evidently (H) satisfies (3) and (4), (H) has property B, and (H) and $\mathcal{R}$(H) are equivalent over the set of all bounded sequences. However, we find that (H) evaluates the unbounded sequence

$$x_n = 1 + (-1)^n i \log (n+2)$$

to 0; and further that (H) evaluates $\{\mathcal{R}(x_n)\}$ to 1, $\{\mathcal{J}(x_n)\}$ to -1, $\{\bar{x}_n\}$ to 2, and the real sequence $\{i\mathcal{J}(x_n)\}$ to the imaginary value $-i$. Finally, $\mathcal{R}$(H) evaluates $\{x_n\}$ to 1.

This example shows that (4) is not sufficient to ensure mutual consistency of a regular (G) and its associated real transformation $\mathcal{R}$(G).

The condition (4) is not necessary in order that (G) and $\mathcal{R}$(G) may be equivalent, or that (G) may have properties A and B. In fact, it is easy to show that the regular transformation

$$(\mathrm{J}) \qquad \sigma_n = [i + (-1)^n i] s_{n-1} + [1 - i - (-1)^n i] s_n,$$

which fails to satisfy (4), is equivalent to its associated real transformation $\mathcal{R}$(J) which obviously has properties A and B.

PRINCETON UNIVERSITY

A RELATIONSHIP IN SPHERICAL HARMONICS

BY J. H. WEBB

It is known that a solid spherical harmonic about one set of axes can be expressed as a solid spherical harmonic with constant coefficients about a new set of axes whose origin is displaced from that of the first. A new relationship has been found for making this transformation and a proof is herein given.

Consider the two sets of cartesian coordinates x, y, z and x_1, y_1, z_1, whose origins O and O_1 are separated by a distance h. Let the two axes x and x_1 be coincident, but let the positive sense of the two axes be opposite. Consider the point S which has the two sets of polar coordinates defined by the equations

$$x_1 = \rho_1 \cos\theta_1 = \rho_1\mu_1, \quad x = \rho\cos\theta = \rho\mu,$$
$$y_1 = \rho_1 \sin\theta_1 \sin\phi_1, \quad y = \rho\sin\theta\sin\phi,$$
$$z_1 = \rho_1 \sin\theta_1 \cos\phi_1, \quad z = \rho\sin\theta\cos\phi.$$

The transformation to be obtained may be expressed symbolically as follows:

$$(1) \qquad \sum_{m=0}^{\infty} \rho_1^{-(m+1)} Y_m(\mu_1, \phi_1) = \sum_{m=0}^{\infty} f_m(\rho) Y_m(\mu, \phi),$$

in which Y_m signifies the usual spherical harmonic. Such a transformation has been given previously by B. Datta.* The new relationship which has been found to give the transformation (1) and which appears to have been unnoticed before is

$$(2) \qquad \frac{P_m^n(\mu_1)}{\rho_1^{m+1}} = a_{(m-n)}(1-\mu^2)^{n/2} \frac{\partial^{m-n}}{\partial h^{m-n}} \left(\frac{1}{h^n} \frac{\partial^n}{\partial \mu^n} \frac{1}{\rho_1} \right),$$

in which $a_{(m-n)}$ designates the term $(-1)^{m-n}/(m-n)!$. A proof of this relationship follows.

Relative to the partial differentiation on the right side of (2), ρ_1 is to be expressed in terms of ρ, μ and h. At certain stages of the calculation it will be convenient to express ρ_1 by means of the equation

* Tôhoku Mathematical Journal, vol. 15 (1919), p. 166.

$$\frac{1}{\rho_1} = \frac{1}{h} \sum_{k=0}^{\infty} \left(\frac{\rho}{h}\right)^k P_k(\mu), \tag{3}$$

and at others by the equation

$$\frac{1}{\rho_1} = [\rho^2 - 2\rho h\mu + h^2]^{-1/2} = R^{-1}. \tag{4}$$

Differentiating ρ_1^{-1} as given by equation (4), with respect to μ, n times and dividing by $(\rho h)^n$, gives

$$(\rho h)^{-n} \frac{\partial^n \rho_1^{-1}}{\partial \mu^n} = 1 \cdot 3 \cdots (2n-1) R^{-(2n+1)},$$

and this may be simplified so as to read

$$h^{-n} \frac{\partial^n \rho_1^{-1}}{\partial \mu^n} = c(n) \frac{\rho^n}{\rho_1^{2n+1}}, \tag{5}$$

in which $c(n)$ designates the term $(2n)!/(2^n n!)$. Furthermore by the law of sines, (see Fig. 1),

$$(1-\mu^2)^{n/2} = (1-\mu_1^2)^{n/2} \left(\frac{\rho_1}{\rho}\right)^n. \tag{6}$$

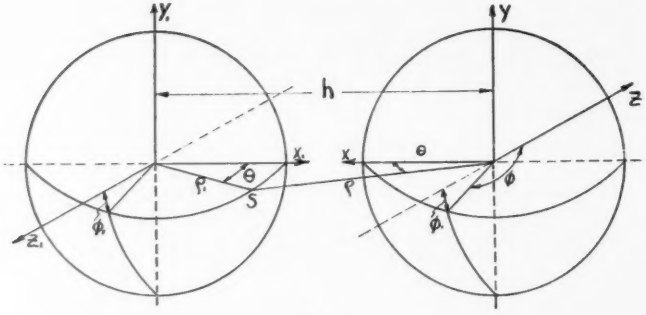

FIG. 1

Substituting the above value in the right side of equation (2) and replacing the left side by its value from the defining equation

$$(7) \qquad \frac{P_m^n(\mu_1)}{\rho_1^{m+1}} = \frac{(1-\mu_1^2)^{n/2}}{\rho_1^{m+1}} \frac{\partial^n P_m(\mu_1)}{\partial \mu_1^n},$$

we obtain in place of equation (2) the equation

$$(2') \qquad \frac{(1-\mu_1^2)^{n/2}}{\rho_1^{m+1}} \frac{\partial^n P_m(\mu_1)}{\partial \mu_1^n} = a_{(m-n)}(1-\mu_1^2)^{n/2} c(n) \frac{\partial^{m-n}}{\partial h^{m-n}} \rho_1^{-(n+1)},$$

and this may be written as

$$\frac{1}{\rho_1^{m+n+1}} \frac{\partial^n P_m(\mu_1)}{\partial \mu_1^n} = a_{(m-n)} c(n) \frac{\partial^{m-n}}{\partial h^{m-n}} \rho_1^{-(2n+1)}.$$

It has been shown* that the following equation is true:

$$(8) \qquad \frac{\partial^n P_m(\mu_1)}{\partial \mu_1^n} = \frac{c(m)}{(m-n)!}\left[\mu_1^{m-n} - \frac{(m-n)(m-n-1)}{2[2n+2(m-n)-1]}\mu_1^{m-n-2}\right.$$
$$\left. + \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2\cdot 4[2n+2(m-n)-1][2n+2(m-n)-3]}\mu_1^{m-n-4} - \cdots\right].$$

Therefore to establish equation (2), it is only necessary to show that the equation

$$(9) \qquad (-1)^{m-n} c(n) \frac{\partial^{m-n}}{\partial h^{m-n}} \rho_1^{-(2n+1)} = \frac{1}{\rho_1^{m+n+1}} \frac{\partial^n P_m(\mu_1)}{\partial \mu_1^n}$$

is true. This can be shown by an induction argument applied to the parameter $(m-n)$. First note that the proposed formula is true for $(m-n)=1$. Then $m=n+1$ and the proposed formula reads

$$(10) \qquad c(n) \frac{\partial}{\partial h} R^{-(2n+1)} = c(n+1) \frac{\mu_1}{\rho_1^{2n+2}},$$

or

$$R^{-(2n+3)}(h-\rho\mu) = \mu_1 \rho_1^{-(2n+2)}.$$

Replacing $(h-\rho\mu)$ by $\rho_1\mu_1$, we easily verify that this equation is an identity. Therefore, let the validity of the proposed equation, for index $(m-n)$, be assumed and show that it holds when $(m-n)$ is replaced by $(m-n+1)$. That is, let the following equation be assumed:

* Byerly, *Fourier Series and Spherical Harmonics*, 1893, p. 197.

$$(-1)^{m-n}\frac{\partial^{m-n}\rho_1^{-(2n+1)}}{\partial h^{m-n}} = (2n+1)(2n+3)\cdots$$

$$\cdot\{2n+2(m-n)-1\}\Big[\frac{(\rho_1\mu_1)^{m-n}}{\rho_1^{[2n+2(m-n)+1]}}$$

$$(11)\qquad -\frac{(m-n)(m-n-1)}{2[2n+2(m-n)-1]}\,\frac{(\rho_1\mu_1)^{m-n-2}}{\rho_1^{[2n+2(m-n)-1]}}$$

$$+\frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2\cdot 4[2n+2(m-n)-1][2n+2(m-n)-3]}$$

$$\cdot\frac{(\rho_1\mu_1)^{m-n-4}}{\rho_1^{[2n+2(m-n)-3]}}-\cdots\Big].$$

Equation (11) differs formally from equation (9), whose validity is to be established, in that the coefficient of the square bracket on the right side has been modified in form. The purpose of this modification is to bring into explicit evidence the parameter $(m-n)$ to which the induction process applies. Replace $\rho_1\mu_1$ in (11) by its value, $(h-\rho\mu)$, differentiate once with respect to h and replace $(h-\rho\mu)$ in the resulting equation by $(\rho_1\mu_1)$. The resulting equation is

$$(-1)^{m-n+1}\frac{\partial^{m-n+1}}{\partial h^{m-n+1}}\rho_1^{-(2n+1)} = (2n+1)(2n+3)\cdots$$

$$\cdot\{2n+2(m-n)+1\}\Big[\mu_1^{m-n+1}$$

$$(12)\qquad -\frac{(m-n+1)(m-n)}{2[2n+2(m-n)+1]}\mu_1^{m-n-1}$$

$$+\frac{(m-n+1)(m-n)(m-n-1)(m-n-2)}{2\cdot 4[2n+2(m-n)+1][2n+2(m-n)-1]}\mu_1^{m-n-3}-\cdots\Big].$$

By comparing equation (12) with (11), it is seen that wherever $(m-n)$ occurs in (11), $(m-n+1)$ occurs in (12), thus proving the validity of equation (11).

If now ρ_1^{-1} be replaced in equation (2) by its value as given by equation (3), the result is

$$(13)\quad \frac{P_m^n(\mu_1)}{\rho_1^{m+1}} = a_{(m-n)}(1-\mu^2)^{n/2}\frac{\partial^{m-n}}{\partial h^{m-n}}\frac{1}{h^n}\frac{\partial^n}{\partial \mu^n}\sum_{k=0}^{\infty}\frac{\rho^k}{h^{k+1}}P_k(\mu).$$

Differentiating with respect to μ and h, we obtain

$$(14)\quad \frac{P_m^n(\mu_1)}{\rho_1^{m+1}} = \sum_{k=n}^{\infty}\frac{(m+k)!}{(m-n)!(n+k)!}\frac{\rho^k}{h^{m+k+1}}P_k^n(\mu) = \sum_{k=n}^{\infty}\alpha_{knm}.$$

For the special case $n=0$, (14) reduces to the form*

$$(15)\quad \frac{P_m(\mu_1)}{\rho_1^{m+1}} = \sum_{k=0}^{\infty}\frac{(m+k)!}{m!k!}\frac{\rho^k}{h^{m+k+1}}P_k(\mu) = \sum_{k=0}^{\infty}\alpha_{k0m}.\dagger$$

The formulas (14) and (15) furnish a means of obtaining the desired transformation (1). In fact the left member of equation (1), written in full, is

$$(16)\quad \sum_{m=0}^{\infty}A_{0m}\sum_{k=0}^{\infty}\alpha_{k0m} + \sum_{n=1}^{m}[A_{nm}\cos n(180^\circ-\phi) + B_{nm}\sin n(180^\circ-\phi)]\sum_{k=n}^{\infty}\alpha_{knm}.$$

By an interchange of the indices m and k, and by noting that $\cos n(180-\phi)=(-1)^n\cos n\phi$ and $\sin n(180^\circ-\phi)=\sin n\phi$, (16) can be thrown into the desired form,

$$(17)\quad \sum_{m=0}^{\infty}\left[\sum_{k=0}^{\infty}A_{0k}\alpha_{m0k} + \sum_{n=1}^{m}\sum_{k=n}^{\infty}(A_{nk}\cos n\phi + B_{nk}\sin n\phi)\alpha_{mnk}\right],$$

and it can be shown that such a rearrangement of terms in the triple series is legitimate providing the A_{nk} and the B_{nk} are bounded. It is to be noted that $A_{nk}=(-1)^nA_{nk}$.

In conclusion, I should like to express my thanks to Professor Warren Weaver of the University of Wisconsin for his helpfulness and interest in the above work.

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* See Whittaker and Watson, *Modern Analysis*, 4th edition, p. 400, ex. 7.
† See B. Datta, loc. cit., p. 170.

CRITERIA FOR THE SOLUTION OF A CERTAIN QUADRATIC DIOPHANTINE EQUATION

BY R. G. ARCHIBALD

1. *Introduction.* The interesting diophantine equation,

$$ax^2 + by^2 + cz^2 + du^2 = 0, \tag{1}$$

in which a, b, c, d are integers, all different from zero, and x, y, z, u are the unknowns, has already been treated in the literature. It is, however, desirable to have a complete treatment and definite statement of criteria for solvability directly applicable to a given equation.

In 1884 A. Meyer* stated, though somewhat obscurely, necessary and sufficient conditions for the solvability of equation (1), but his proof, as well as P. Bachmann's† treatment, is restricted to the case in which a, b, c, d are all odd integers. In 1930 an account of this equation was given by L. E. Dickson;‡ but, as he points out himself, his work is incomplete in the case in which exactly two of the coefficients are even integers and $abcd/4 \equiv 5 \pmod 8$. Recently a paper§ by L. J. Mordell has appeared in which a complete and independent derivation of conditions for solvability is given. His conditions, however, are not always directly applicable to a given equation: more explicitly, his condition II may not be satisfied by a given equation and yet the equation may possess a solution. According to the scheme there presented, restrictions are placed on the values of the unknowns, and it may happen that a set of diophantine equations have to be tested to determine whether the given one is solvable or not. This is best illustrated by an example.

His method is not directly applicable to the equation

$$110x^2+770y^2-z^2-u^2=0,$$

which is solvable for $x=2$, $y=1$, $z=11$, $u=33$). We enquire,

* Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, vol. 29 (1884), pp. 209–222.

† *Zahlentheorie*, Part IV, *Die Arithmetik der Quadratischen Formen*, I, pp. 259–266.

‡ *Studies in the Theory of Numbers*, pp. 70–76.

§ Journal für Mathematik, vol. 164 (1931), pp. 40–49.

without loss of generality, whether this equation has a solution such that x, y, z, u have no common factor.

Since Mordell's condition II is not satisfied, we may gather that either the equation has no (non-trivial) solution or both z and u have either the factor 5 or the factor 11, or both. If z and u both have the factor 5, we are led to the equation $22x^2+154y^2-5z_1^2-5u_1^2=0$, for which again condition II is not satisfied. (By our assumption that x, y, z, u have no common factor we know that x and y are not divisible by 5.)

If the equation just mentioned has a solution, both z_1 and u_1 must have the factor 11; consequently we are led to the equation $2x^2+14y^2-55z_2^2-55u_2^2=0$. Here, again, condition II is not satisfied. Thus, the original equation either possesses no solution, or z and u do not have the factor 5.

Next, we may suppose that both z and u have the factor 11 and obtain the equation $10x^2+70y^2-11z_6^2-11u_6^2=0$, for which the condition II is satisfied. Hence the original equation, as tested by Mordell's criterion, has a solution.

It is our object here to obtain, as a special case and application of important considerations* of H. Hasse, necessary and sufficient conditions, which can be directly applied to a given equation, for the solvability in integers, not all zero, of equation (1) with integral (rational) coefficients a, b, c, d, all different from zero.

2. *Necessary and Sufficient Conditions for Solvability. Preliminary Conditions.* Without loss of generality we may assume that†

(i) Each of a, b, c, d is without a squared factor >1.

(ii) No three of a, b, c, d have a common factor >1.

Necessary and sufficient conditions‡ for the solvability of equa-

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* *Über die Darstellbarkeit von Zahlen durch quadratische Formen im Körper der rationalen Zahlen*, Journal für Mathematik, vol. 152 (1923), pp. 129–148. This paper was brought to the writer's attention by Hasse.

† See, for example, Meyer, loc. cit., pp. 209–210, or Dickson, loc. cit., pp. 68–71.

‡ We employ the notation used by Meyer; namely,

$$a = (a, b)(a, c)(a, d)\alpha, \qquad b = (a, b)(b, c)(b, d)\beta,$$
$$c = (a, c)(b, c)(c, d)\gamma, \qquad d = (a, d)(b, d)(c, d)\delta,$$

where (f, g) denotes the greatest common divisor of f and g.

tion (1), with a, b, c, d non-zero integers, satisfying the above preliminary conditions are the following:

I a, b, c, d are not all of the same sign.

II (i) $-(a, c)(a, d)(b, c)(b, d)\gamma\delta$ is a quadratic residue of every odd prime p dividing either (a, b) or (c, d) such that at the same time Legendre's symbol $(\alpha\beta\gamma\delta/p) = +1$;

(ii) $-(a, b)(a, d)(b, c)(c, d)\beta\delta$ is a quadratic residue of every odd prime p dividing either (a, c) or (b, d) for which $(\alpha\beta\gamma\delta/p) = +1$;

(iii) $-(a, b)(a, c)(b, d)(c, d)\beta\gamma$ is a quadratic residue of every odd prime p dividing either (a, d) or (b, c) for which $(\alpha\beta\gamma\delta/p) = +1$.

III Either

(i) $abcd \equiv 2, 3, 5, 6, 7 \pmod 8$;

or

(ii) $abcd \equiv 1$ and $a+b+c+d \equiv 0 \pmod 8$;

or

(iii) $abcd \equiv 4 \pmod 8$, and, if a and b are even and c and d odd, either $abcd/4 \equiv 3, 5, 7 \pmod 8$, or $abcd/4 \equiv 1 \pmod 8$ and

$$\frac{a}{2} + \frac{b}{2} + c + d \equiv \frac{(cd)^2 - 1}{2} \pmod 8.$$

3. *Application of Hasse's Work.* By the *discriminant* of a quadratic form we mean the determinant $D = |a_{ij}|$ of its set of coefficients. We shall consider only forms which have non-zero discriminant. By the *rational kernel* of a number m, different from zero, in the field $K(1)$ of all rational numbers, we mean the uniquely determined reduced number obtained by removing all rational squared factors from m. Similarly, the *p-adic kernel* of a number $m(\neq 0)$ in the field* $K(p)$ of p-adic numbers is the reduced number obtained by removing from m all squared factors occurring in the field $K(p)$. The rational kernels are integral positive or negative numbers without squared factors. Rationally equivalent forms have the same rational kernel for their discriminants, and p-adically equivalent forms have the same p-adic kernel for their discriminants. The p-adic kernels, say d_p, of the discriminant of one and the same form in the field

* See K. Hensel, *Zahlentheorie*, 1913, Chapter 6.

$K(1)$ are at the same time the p-adic kernels of the rational kernel, d, of the discriminant. We speak of the *invariants* d and d_p.

Consider a ternary quadratic form f. We define for a form equivalent, in the algebraic sense, to f and without cross-product terms (which always exists), let us say $f_0 = by^2+cz^2+du^2$, the quantity $\check{c}_p$ as the generalized Hilbert* norm residue symbol

$$\check{c}_p = \check{c}_p(f_0) = \left(\frac{-bc, -bd}{p}\right).$$

This quantity $c_p(f_0)$ is an invariant with respect to p-adic transformations and can be denoted by $\check{c}_p(f)$. We consider now Hasse's theorem† that a quaternary quadratic form (with rational coefficients) of invariant d represents zero rationally if and only if in any form whatever of the special type $ax^2+\phi(y, z, u)$ rationally equivalent to it, $\check{c}_p(\phi) = +1$ whenever the generalized symbol‡ of Legendre (d/p) has the value $+1$, this holding true for every finite prime p and also§ for $p = p_\infty$. For our purpose let $f = ax^2+by^2+cz^2+du^2$ and $\phi = by^2+cz^2+du^2$, whence

$$\check{c}_p(\phi) = \left(\frac{-bc, -bd}{p}\right).$$

In the form $ax^2+by^2+cz^2+du^2$, with integral rational coefficients, let us assume $abcd \neq 0$ and the preliminary conditions (i) and (ii) to hold. Hence, the invariant d is $\alpha\beta\gamma\delta$.

Our equation (1), *then, with* $abcd \neq 0$ *and with the preliminary conditions* (i) *and* (ii) *satisfied, is solvable in rational numbers* (*or, what is equivalent, in integers*) *if, and only if, for every finite prime* p *and for* $p = p_\infty$, *the Hilbert symbol*

$$\left(\frac{-bc, -bd}{p}\right) = +1 \tag{2}$$

when the generalized Legendre symbol $((\alpha\beta\gamma\delta)/p) = +1$.

Now we have

$$-bc = -(b, c)^2(a, b)(b, d)(a, c)(c, d)\beta\gamma,$$

$$-bd = -(b, d)^2(a, b)(b, c)(a, d)(c, d)\beta\delta.$$

* Hensel, loc. cit., Chapter 12, §§4–6.

† Hasse, loc. cit., p. 143.

‡ Hasse, loc. cit., p. 134.

§ See Hensel, loc. cit., Chapter 12, §1.

In the first place, consider the field $K(p_\infty)$, that is, the case $p=p_\infty$. Now $(\alpha\beta\gamma\delta/p_\infty)=+1$ when and only when $\alpha\beta\gamma\delta>0$. But when $p=p_\infty$, $((-bc,-bd)/p_\infty)=+1$ if at least one of $-bc$, $-bd$ is positive. Hence, whenever $\alpha\beta\gamma\delta>0$, either β and γ must be of opposite sign or β and δ are of opposite sign; and whenever $\alpha\beta\gamma\delta<0$, the condition (2) is obviously satisfied. Hence, in this case the condition (2) is equivalent to saying that not all of a, b, c, d are of the same sign (our condition I).

Secondly, let p be an *odd* prime. Now, since a squared factor cannot occur in $\alpha\beta\gamma\delta$, and since, if p itself divided $\alpha\beta\gamma\delta$, then $(\alpha\beta\gamma\delta/p)\neq+1$, we need only consider primes not dividing $\alpha\beta\gamma\delta$; that is, only odd primes not dividing $abcd$, and those odd primes dividing exactly two of a, b, c, d.

(i) Let p be an odd prime dividing either (a, b) or (c, d). Then*

$$\left(\frac{-bc,-bd}{p}\right)=\left(\frac{-(b,d)(a,c)(b,c)(a,d)\gamma\delta}{p}\right).$$

It follows, therefore, that in this case our condition states that $-(b, d)(a, c)(b, c)(a, d)\gamma\delta$ is a quadratic residue of every odd prime factor p of either (a, b) or (c, d) such that the relation $(\alpha\beta\gamma\delta/p)=+1$ holds (our condition II (i)).

(ii) Let p be an odd prime dividing either (a, c) or (b, d). Then

$$\left(\frac{-bc,-bd}{p}\right)=\left(\frac{-(a,b)(b,c)(a,d)(c,d)\beta\delta}{p}\right).$$

It follows, therefore, that in this case, our condition states that $-(a, b)(b, c)(a, d)(c, d)\beta\delta$ is a quadratic residue of every odd prime factor p of either (a, c) or (b, d) such that the relation $(\alpha\beta\gamma\delta/p)=+1$ holds (our condition II (ii)).

(iii) Let p be an odd prime dividing either (a, d) or (b, c). Then

$$\left(\frac{-bc,-bd}{p}\right)=\left(\frac{-(a,b)(b,d)(a,c)(c,d)\beta\gamma}{p}\right).$$

Hence our condition states that $-(a, b)(b, d)(a, c)(c, d)\beta\gamma$ is a quadratic residue of every odd prime factor p of either (a, d) or (b, c) such that $(\alpha\beta\gamma\delta/p)=+1$ (our condition II (iii)).

* Hensel, loc. cit., p. 317.

Thirdly, let p be an odd prime dividing *none* of a, b, c, d. Then $((-bc, -bd)/p)$ is automatically equal to $+1$.

Fourthly, let p be the prime $p=2$.

(i) Let 2 be a factor of one only of a, b, c, d. The condition is vacuously satisfied since $(\alpha\beta\gamma\delta/2) \neq +1$. Hence, if $abcd \equiv 2$ or 6 (mod 8), no further condition is required (part of condition III (i)).

(ii) Let 2 be a factor of precisely two of a, b, c, d. Without loss of generality, let us suppose that a and b are even and that c and d are odd (that is, $(a/2)(b/2)cd \equiv 1, 3, 5, 7 \pmod 8$). We desire to have $((-bc, -bd)/2) = +1$ whenever $(\alpha\beta\gamma\delta/2) = +1$. But $(\alpha\beta\gamma\delta/2) = +1$ if and only if $\alpha\beta\gamma\delta \equiv 1 \pmod 8$; that is, if and only if $(a/2)(b/2)cd \equiv 1 \pmod 8$. Hence, employing the usual methods of evaluating the symbol $((f, g)/2)$, we desire

$$\frac{\left\{-(b,c)^2\frac{(a,b)}{2}(b,d)(a,c)(c,d)\beta\gamma\right\}^2-1}{8}+\frac{\left\{-(b,d)^2\frac{(a,b)}{2}(b,c)(a,d)(c,d)\beta\delta\right\}^2-1}{8}+\frac{\left\{-(b,c)^2\frac{(a,b)}{2}(b,d)(a,c)(c,d)\beta\gamma-1\right\}}{2}\cdot\frac{\left\{-(b,d)^2\frac{(a,b)}{2}(b,c)(a,d)(c,d)\beta\delta-1\right\}}{2}$$

to be *even* when $\alpha\beta\gamma\delta \equiv 1 \pmod 8$. Now, since for two *odd* integers A and B,

$$\frac{A^2-1}{2}+\frac{B^2-1}{2} \equiv \frac{(AB)^2-1}{2} \pmod{32},$$

we have finally, remembering that the square of an odd number is of the form $8M+1$ and that $(a/2)(b/2)cd \equiv 1 \pmod 8$ (whence $(b/2)c+1 \equiv (a/2)d+d^2 \pmod 8$),

$$d^2 \cdot \frac{(cd)^2 - 1}{2} + cd + \frac{b}{2} d + \frac{a}{2} d + d^2 \equiv 0 \pmod 8$$

when $\alpha\beta\gamma\delta \equiv 1 \pmod 8$, that is, when $(a/2)(b/2)cd \equiv 1 \pmod 8$. We note that

$$-d \cdot \frac{(cd)^2 - 1}{2} \equiv \frac{(cd)^2 - 1}{2} \pmod 8,$$

and obtain on simplification the condition that

$$\frac{a}{2} + \frac{b}{2} + c + d \equiv \frac{(cd)^2 - 1}{2} \pmod 8$$

when $(a/2)(b/2)cd \equiv 1 \pmod 8$. If $(a/2)(b/2)cd \equiv 3, 5, 7 \pmod 8$, that is, $\alpha\beta\gamma\delta \equiv 3, 5, 7 \pmod 8$, we saw that $(\alpha\beta\gamma\delta/2) \neq +1$. Hence we have our condition III (iii).

(iii) Finally, let each of a, b, c, d be odd. Then

$$\left(\frac{-bc, -bd}{2}\right) = (-1)^s,$$

where

$$s = \frac{-(b, c)^2(a, b)(b, d)(a, c)(c, d)\beta\gamma - 1}{2} \cdot \frac{-(b, d)^2(a, b)(b, c)(a, d)(c, d)\beta\delta - 1}{2}.$$

Hence we desire s to be even whenever $(\alpha\beta\gamma\delta/2) = +1$. But $(\alpha\beta\gamma\delta/2) = +1$ when and only when $abcd \equiv 1 \pmod 8$. The condition is therefore vacuously satisfied when $abcd \equiv 3, 5, 7 \pmod 8$ (part of condition III (i)). Thus, when $abcd \equiv 1 \pmod 8$, we desire

$$\{(b, c)^2(a, b)(b, d)(a, c)(c, d)\beta\gamma + 1\} \cdot \{(b, d)^2(a, b)(b, c)(a, d)(c, d)\ \beta\delta + 1\} \equiv 0 \pmod 8.$$

That is, $cd+bc+bd+abcd \equiv 0 \pmod 8$; or, in other words, since $abcd \equiv 1 \pmod 8$, $a+b+c+d \equiv 0 \pmod 8$. This is our condition III (ii).

All possible values of p have been considered. The set of conditions I, II, and III is therefore necessary and sufficient for the solvability of equation (1) in integers.

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A RATIONAL QUINTIC SURFACE HAVING NO DOUBLE CURVE

BY A. R. WILLIAMS

Montesano in his extensive researches on rational quintic surfaces has noted briefly one such surface obtained by applying a quadratic transformation to the quartic surface known as Nöther's third type.* This quintic surface can be obtained by transformation of a rational sextic surface, due also to Montesano. The plane representation of the quintic which results is the same as that given by Montesano; but, aside from the new method of derivation, there are certain features of the surface and the plane representation not mentioned by him, which it is the purpose of this note to consider.

We start with the rational sextic surface that has a quadruple point Q at which are concurrent 3 coplanar double lines and a triple line, which of course does not lie in the plane of the double lines. If we apply a quadratic transformation of the first type whose fixed conic is the triple line and one of the double lines, and whose fixed point is on another of the double lines, we obtain a rational quintic, noted by Montesano,† which has two consecutive skew double lines, and which I have described in a previous paper.‡ If, however, we apply to the sextic a quadratic transformation whose fixed conic consists of two of the double lines and whose fixed point is on the triple line we obtain another rational quintic which is the subject of this paper.

The plane system of the rational sextic surface is the web of curves of order 9 having in common 8 triple points $A_1, \cdots, A_8$ and 3 simple points B_1, B_2, B_3. This set of curves may be designated as the web of nonics $8A^3B_1B_2B_3$. The image of the triple line k is the plane sextic $8A^2B_1B_2B_3$. The image of a double line d_i is the cubic $8AB_i$. To B_i corresponds the residual line on the sextic surface in the plane of k and d_i. The tangent cone to the sextic at the quadruple point Q consists of the 4 planes kd_1, kd_2, kd_3, and $d_1d_2d_3$.

* Rendiconti di Napoli, vol. 40 (1901), p. 100.

† Ibid., vol. 46 (1907), p. 66.

‡ This Bulletin, vol. 34 (1928), p. 761.

To the elements of direction on the surface at Q in the plane kd_i correspond the elements of direction at B_i' in the image plane. The point B_i' lies on the cubic $8AB_i$ and is the conjugate of B_i in I_{17}, the involution determined by the sextics $8A^2$. All sextics of this web that pass through a general point of the plane pass also through another point determined by the first. To the surface elements at Q in the plane of the double lines correspond the elements about A_0, the ninth base point of the pencil of cubics of $8A$. If we now apply a quadratic transformation whose fixed conic is d_1d_3 and whose fixed point is a general point O on the triple line k, the resulting surface is of order 5. For it loses the planes kd_1 and kd_3 each twice, and the plane $d_1d_2d_3$ three times. A general ray through Q meets the sextic twice elsewhere. Its transform is another ray through Q. Therefore the quintic has a triple point at Q. Similarly a general ray through O meets the sextic in 3 other points. Such a ray is invariant as a whole; and hence the quintic has a double point at O. The image of the curve of intersection of a general quadric with the sextic is a curve of order 18, $8A^6B_1^2B_2^2B_3^2$. Since the homoloidal quadrics contain d_1 and d_3 and O on the triple line k, the image of the intersection of such a quadric with the sextic is of order 12, and has quadruple points at $A_1, \cdots, A_8$, a double point at B_2, and simple points at B_1 and B_3 and at E_1, E_2, E_3, the three points on the sextic image of the triple line that correspond to O. But this curve of order 12 fulfills 2 more conditions in that it is tangent at B_2' to the cubic $8AB_2B_2'$. For a homoloidal quadric has d_1 and d_3 for generators, and hence its intersection with the sextic surface is tangent to d_2 at Q. Therefore the plane sections of the quintic surface have as images the web of duodecimics $8A^4B_2^2B_2'B_2''B_1B_3E_1E_2E_3$. They may be called the base curves. The point B_2'' is consecutive to B_2' on the cubic $8AB_2B_2'$. The 3 tangent planes to the sextic at O give 2 lines, e_1, e_2, e_3 through Q in the plane $d_1d_2d_3$. They correspond to E_1, E_2, E_3 in the new plane system. The residual lines on the sextic in the planes kd_1 and kd_3 give d_1 and d_3 as simple lines on the quintic. They correspond to B_1 and B_3 in the plane system of the latter. The trace of the quintic in the plane of d_1d_3 is therefore the 5 simple lines d_1, d_3, e_1, e_2, e_3 through Q. We shall call this plane σ. The residual line on the sextic in the plane of the triple line and d_2 gives a conic on the quintic in the same plane, passing through O and Q

and tangent to the plane σ at Q. The image of this conic is B_2. Its plane π is tangent to the quintic along k, the residual intersection being the conic just mentioned. A general plane of the pencil k meets the sextic surface in a residual cubic which has the intersection of its plane with σ for inflectional tangent at Q. By the transformation this cubic becomes a quartic in the same plane, tangent to k at O and having a tacnode at Q, the branches being tangent to σ. Therefore the line k, triple on the sextic, is a simple line of the quintic, joining the nodes Q and O. Also the tangent cone to the quintic at Q is the plane σ taken twice and the plane π. At O the tangent cone is π taken twice. The image of the triple point Q of the quintic surface is the same plane sextic that corresponds to the triple line of the sextic surface, that is, the sextic $8A^2B_2B_2'B_1B_3E_1E_2E_3$. The last 3 points are the residual intersections of the sextic with a net of nonics $8A^2B_1B_2B_3E_1(E_2E_3)$. To this net of nonics corresponded on the sextic surface the plane sections through O. Since by the quadratic transformation such a plane is invariant as a whole, the nonics of this net are the images of the plane sections of the quintic surface by planes through O. Such a plane cuts the sextic in a plane curve having a triple point at O, and double points on d_1, d_2, d_3. Considering the quadratic transformation imposed in such a plane by the general transformation we see that the plane section of the sextic surface becomes a plane quintic with a tacnode at O. The unode at O is therefore a tacnode of the quintic surface. Its image is the cubic $8AB_2B_2'$.

Among the ∞^3 homoloidal quadrics which effect the transformation are the ∞^2 cones whose vertex is Q and which contain the lines d_1, d_3, and k. These cones intersect the sextic surface in skew quintic curves whose images are the sextics of the net $8A^2B_2B_2'$. The cones become planes through Q. Therefore the sextics of the net just mentioned are the images of sections of the quintic surface by planes through its triple point Q. To sections by planes through d_1 correspond the ∞^1 sextics of the net that pass through B_1. Similarly for sections through d_3, e_1, e_2, e_3. The section of the quintic by a general plane through Q has at Q a triple point composed of three linear branches two of which have contact, the tangent lying in σ. The tangent to the other branch lies in π. To sections by planes through k, the line joining the nodes, correspond the cubics of the pencil $8A$.

Twelve of these have a node, and the corresponding sections are tangent to the surface. In particular, to the sections by the planes kd_1, kd_3, ke_1, ke_2, ke_3 correspond the cubics of the pencil determined by B_1, B_3, E_1, E_2, E_3. To A_0, the ninth base point of the pencil of cubics, corresponds the direction of k at O.

In what follows it will be convenient to refer to the image of the triple point Q, that is, the sextic $8A^2B_2B_2'B_1B_3E_1E_2E_3$, as s; and to the image of the unode O, that is, the cubic $8AB_2B_2'$, as ϕ. The number of base curves of a general pencil that have an additional node, that is, the class of the surface, is 38. For we have in the first place $3(12-1)^2$ less 39 for each quadruple point, and less 7 for the double point B_2; that is, 44. But a general pencil of planes contains one section through each node. To the section through O corresponds in the pencil of plane base curves a nonic of the net above described and ϕ. This nonic meets ϕ at B_2 and in two more points which are to be deducted. To the section through Q corresponds a sextic of the net $8A^2B_2B_2'$. The two additional points in which it meets s are to be deducted.

But a further reduction of 2 is necessary on account of this degenerate duodecimic. The curves of the pencil have a common tangent at B_2'. When a pencil of curves have a common tangent at a base point the curve of the pencil that has a double point there counts for 2, or reduces by 2 the number of curves of the pencil that have an additional node elsewhere. In the present case this is the degenerate curve composed of s and the variable sextic. Hence the class of the quintic is 38. The Jacobian of a general net of base curves contains the factors s and ϕ. For the net contains two pencils of degenerate duodecimics, one composed of s and a variable sextic, the other of ϕ and a variable nonic. If we remove these factors, it appears that the image of the curve of contact of a general tangent cone is of order 24 and has 8-fold points at $A_1, \cdots, A_8$, a triple point at B_2, and simple points at B_2', B_1, B_3, E_1, E_2, E_3. Of the intersections of two such curves 527 are obvious at the base points. The class of the surface accounts for 38, leaving 11 to be found. We must, in fact, account for 7 more intersections of such a curve with s and 4 more with ϕ. Now it can be shown independently that it (the curve of order 24) is tangent to s at B_2' and meets s in 6 other fixed points, and ϕ in 4 other fixed points, apart from the base points. This accounts for the remaining 11 intersections of two

such curves, and verifies the class of the surface. A brief description of the method of establishing this result is desirable.

Through any point P of space pass a pencil of nodal sections whose axis is PQ, a similar pencil whose axis is PO, and a unique section by the plane POQ. We have seen that the image of the last section is a cubic ψ of the pencil $8A$. To a plane section through PQ corresponds a sextic s' of the net $8A^2B_2B_2'$. To a plane section through PO corresponds a nonic n of the net $8A^3B_1B_2B_3E_1E_2E_3$. Since any three linearly independent curves of a net determine it and its jacobian, we may use to determine a general net of base curves ss', $s\phi\psi$ and $n\phi$. For a given net ψ is unique; but s' (and similarly n) is any one of a pencil. Moreover s' can be written $\alpha s+\beta\phi\psi+\gamma\phi^2$. We then find that $zJ=12s\phi zj$, where J is the jacobian, and j is of order 24, and is the image of the curve of contact of the corresponding tangent cone.

$$\begin{aligned} zj = n[&s'\psi s_x\phi_y + s'\phi s_x\psi_y + \phi\psi s_y s_x' + s\phi s_x'\psi_y - s'\psi s_y\phi_x - s'\phi s_y\phi_x \\ &- \phi\psi s_x s_y' - s\phi s_y'\psi_x - ss'\phi_x\psi_y + ss'\phi_y\psi_x] - ss'[\psi n_x\phi_y \\ &+ \phi n_x\psi_y - \psi n_y\phi_x - \phi n_y\psi_x] + s\phi\psi[n_x s_y' - n_y s_x'].\end{aligned}$$

Putting $\phi=\psi=0$ we see that this expression does not vanish; and therefore the curve of order 24 does not pass through A_0, the ninth base point of the pencil of cubics $8A$. We wish to show that j is tangent to s at B_2'. Since j vanishes at B_2', the values of $(zj)_x$, $(zj)_y$, and $(zj)_z$ at B_2' are the coordinates of the tangent to j at that point. Since ϕ, s, and j vanish at B_2', the only terms in the three partial derivatives of zj that do not vanish at B_2' will arise from the terms $n\psi[s's_x\phi_y+\phi s_y s_x' - s's_y\phi_x-\phi s_x s_y']$. Omitting the value of $n\psi$ at B_2', which is a common factor in the derivatives, we have, at B_2',

$$\begin{aligned}(zj)_x:(zj)_y:(zj)_z &= s_x(s_x'\phi_y - s_y'\phi_x) \\ &: s_y(s_x'\phi_y - s_y'\phi_x): [s_z'(s_x\phi_y - s_y\phi_x) - \phi_z(s_x s_y' - s_y s_x')].\end{aligned}$$

Since $s'=\alpha s+\beta\phi\psi+\gamma\phi^2$, and since ϕ vanishes at B_2', the last expression at B_2' reduces to $\alpha s_z(s_x\phi_y-s_y\phi_x)$. And $s_x'\phi_y-s_y'\phi_x$ at that point reduces to $\alpha(s_x\phi_y-s_y\phi_x)$. Hence j is tangent to s at B_2'. We wish to show further that j meets s in 6 fixed points and n in 4 fixed points apart from the base points. In fact j meets s in the 6 points that are invariant under I_{17}. For n meets s only

at the base points. Hence where zj (and therefore j) meets s apart from the base points we have

$$s'\psi s_z\phi_y + s'\phi s_z\psi_y + \phi\psi s_y s_z' - s'\psi s_y\phi_z - s'\phi s_y\psi_z - \phi\psi s_z s_y' = 0.$$

If we consider the jacobian of s', $\phi\psi$, and s, we find as above that it meets s, apart from the base points, where the last equation is satisfied, that is, where j meets s. But s' is a linear function of s, $\phi\psi$, and ϕ^2. So the jacobian of s', $\phi\psi$, and s is simply the locus of invariant points of I_{17} with the factor ϕ^2 added. This locus is of order 9, has triple points at $A_1, \cdots, A_8$ and meets s in 6 other points. This amounts to saying that in any pencil of sextics of the net $8A^2B_2B_2'$ there are 6 curves each tangent to s at one of 6 fixed points. Similarly s' meets ϕ only at the base points, and hence where j meets ϕ aside from those points we have

$$n(\psi s_z\phi_y - \psi s_y\phi_z - s\phi_z\psi_y + s\phi_y\psi_z) - s\psi(n_z\phi_y - n_y\phi_z) = 0.$$

If we consider the jacobian of n, $s\psi$, and ϕ, we find that it meets ϕ aside from the base points, where this equation is satisfied, that is, where j meets ϕ. Now n and $s\psi$ are two linearly independent nonics of the net $8A^2B_1B_2B_3E_1E_2E_3$. But in any pencil of nonics belonging to the web $8A^2B_1B_2B_3$ are 4 curves each tangent to ϕ at one of 4 fixed points, independent of the pencil considered. For the nonics of the web are the images of the plane sections of the rational sextic surface with which we started, and the cubic ϕ is the image of a double line, d_2, of the surface on which there are 4 pinch points. Therefore the jacobian of n, $s\psi$, and ϕ (and hence j) meets ϕ in these 4 points. For the jacobian of u, v, w meets $w=0$ once at each point of contact of a curve of the pencil $u+kv=0$ with $w=0$.*

The facts just established have an interesting geometrical significance. We have seen that the cones whose vertex is Q and which have the generators k, d_1, and d_3 meet the sextic surface in residual skew quintic curves, whose images are the sextics of the net $8A^2B_2B_2'$. By the quadratic transformation these cones become planes through Q meeting the quintic surface in sections whose images are the same net of sextics. In particular the pencil of cones which are tangent along k to a given plane ρ of the pencil k become a pencil of planes whose axis is the intersection

* Hilton, *Higher Plane Curves*, p. 110.

of ρ and σ, the plane $d_1d_2d_3$. A general plane ρ of k meets the sextic surface in a residual cubic whose image is a cubic of the pencil $8AA_0$. The plane ρ is tangent to the sextic at two generally distinct points on k corresponding to the points M and N, conjugate in I_{17}, in which the latter cubic meets the sextic s, the image of the triple line of the sextic surface and of the triple point Q of the quintic surface. Therefore the cones of the pencil tangent to ρ along k meet the sextic surface in space curves which all pass through these 2 points on k corresponding to M and N, and each of which passes simply through Q in a variable direction tangent to the plane kd_2 (called π), which is of course the direction of the second generator of the cone that lies in π.

Likewise, the quadratic transformation gives a pencil of plane sections of the quintic surface, each having a triple point at Q composed of a tacnode corresponding to M and N, the common tangent being the intersection of ρ and σ, and a variable third linear branch tangent to π; the latter being the transform of the element of direction just mentioned, and corresponding to the passage of the image curve (sextic) through B_2'. For 6 positions of ρ, the points of tangency of ρ on k, and the points M and N in the plane representation coincide. All the plane sextics of the pencil are tangent to s at the consecutive points MN. The resulting plane sections of the pencil of the quintic surface have each a triple point at Q composed of a cusp of the second kind and a variable linear branch tangent to π. The common cuspidal tangent is of course the intersection of ρ and σ. One plane of each of these 6 pencils passes through any point P of space.

This accounts for the fact noted above that the jacobian of a general net of base curves passes through the corresponding 6 points on s. One sextic of the plane pencil has a double point at the consecutive points MN. The corresponding plane section of the quintic has an oscnode and a third linear branch at Q. The pencil of cones tangent along k to the plane kd_2 (π) and containing d_1 and d_3 is interesting. The plane π is tangent to the sextic surface at L where the residual line in π meets k. Hence these cones meet the sextic surface in curves which are tangent to π at L and osculate π at Q, the tangent at Q being k. The images of these curves are the sextics of the pencil of the net $8A^2B_2B_2'$ tangent to s at B_2 and B_2'. To the passage of the image curve through B_2' tangent to s corresponds the passage of the space

curve through Q tangent to k. By the quadratic transformation such an element at Q becomes a cusp of the first kind, the cuspidal tangent being the line $\pi\sigma$. Therefore the resulting curves of plane sections of a pencil of the quintic surface have each an ordinary cusp at Q and a linear branch, both tangent to the line $\pi\sigma$.

We have seen that the jacobian of a general net of base curves passes through 6 fixed points on s and is tangent to s at B_2'. Hence the curve of contact of the corresponding tangent cone (of order 20) has at Q 6 linear branches tangent to fixed directions in σ and a cusp whose tangent is the line $\pi\sigma$. The other 5 intersections of this curve with σ take place one on each of the 5 lines of the surface lying in that plane. Hence a general tangent cone has at Q a multiple edge equivalent to 27 double edges and one cuspidal edge. Similarly, since a plane through O is invariant as a whole under the quadratic transformation, the sections of the sextic surface by such planes, whose images are the nonics of the net $8A^3B_1B_2B_3E_1E_2E_3$, become the sections of the quintic surface by the same planes. A general plane through O meets the sextic surface in a curve with a double point on d_2, and the resulting section of the quintic by the same plane has a tacnode at O. The sextic surface has 4 pinch points on d_2 whose image is the cubic $8AB_2B_2'$, called ϕ above. Hence on this cubic are 4 points where the pencil of nonics of the above mentioned net that pass through the point are tangent there to the cubic.

The corresponding sections of the sextic surface have ordinary cusps on d_2, and the resulting sections of the quintic have cusps of the second kind at O instead of tacnodes. One nonic of the plane pencil through such a point has a double point there. The corresponding section of the sextic has a tacnode on d_2, and the resulting section of the quintic has an oscnode at O. We have seen that the jacobian of a general net of base curves of the quintic surface passes through these 4 points on ϕ. Hence the curve of contact of the corresponding tangent cone passes 4 times through O tangent to fixed directions in the plane $kd_2(\pi)$. 9 intersections of this curve with π occur at Q, 3 of them accounted for by the cusp at Q, and the remaining 3 intersections occur on the conic whose image is B_2. Thus a general tangent cone has at O an ordinary quadruple edge equivalent to 6 double edges. This accounts satisfactorily for the Plücker characteristics of the cone. For its genus, which is the genus of j, the

image of its curve of contact, is 26, its order is 20, and its class 38. Hence it has 52 cuspidal edges and the equivalent of 93 double edges. One of the cuspidal edges is at Q. In case of a general quintic surface the tangent cone would have 60 double edges. We have seen that Q accounts for 27 and O for 6, which make up the 33 additional required by the surface in question.

From Q a cone of lines can be drawn tangent elsewhere to the surface. The image of the curve of contact is the locus of invariant points in I_{17}. Similarly the image of the curve of contact of tangents drawn to the surface from O is of order 18, for it loses the factor s. The image of the parabolic curve, that is, the locus of cusps of the base curves, is obtained by deducting one of these base curves from the jacobian of a general net of curves j. Since the intersections of a j with s and ϕ are all fixed, they are factors of the jacobian in question. Hence the image of the parabolic curve is of order 48, and has 16-fold points at $A_1, \cdots, A_8$, and a 4-fold point at B_2, but does not pass through B_2', or any of the other base points. This shows that in a pencil of plane sections through any of the 6 lines of the surface the residual quartic is never tangent to the axis apart from the nodes. The sextics which have double points at $A_1, \cdots, A_8$ but do not pass through B_2 give on the surface octavic curves of genus 2. Such a sextic may be composed, in 28 ways, of the line joining 2 of the points A and the rational quintic having double points at the other six and passing through the first two. The line and the quintic meet 3 times on the locus of invariant points in I_{17}. The 2 additional points in which the line meets s, the image of Q, correspond in I_{17} to the 2 similar points in which s is met by the quintic. Since two points on s that are conjugate in I_{17} give the same direction at Q in σ, we have on the surface 28 pairs of rational quartics, each quartic passing twice through Q and once through O, and the two quartics of a pair being tangent to each other twice at Q. Similarly the sextic may be composed of the conic through five of the points A and the quartic having double points at the other three and passing through the first five. So we get 56 more pairs of such quartics. Finally the rational quartics corresponding to $A_1, \cdots, A_8$ are of this type. With the quartic corresponding to A_i is paired the rational quartic whose image is the sextic that has a triple point at A_i and double points at the other 7 points A. This sextic corresponds to A_i in I_{17}.

Mention may be made of a few slightly specialized forms of the surface. In setting up the plane system we take 8 general points A which determine A_0, and a point B_2 which determines B_2' and the cubic $8AB_2B_2'$, called ϕ. Any proper sextic of the net $8A^2B_2B_2'$ can be taken for s. A proper duodecimic $8A^4B_2^2$ tangent to ϕ at B_2' meets s in $B_1B_3E_1E_2E_3$. Three of the latter points determine the remaining two. If the duodecimic is tangent to s, two of the five corresponding lines in σ coincide and the surface acquires a conic node on the coincident lines. If $E_1E_2E_3$ are taken consecutive on s, three lines in σ become coincident, and the new node is a binode reducing the class by 3. If B_2 is taken on the line joining two of the points $A_1, \cdots, A_8$, this line with s and the quintic which has double points at the remaining six points A and passes through the first two and necessarily through B_2' constitutes the image of a plane section. To the quintic corresponds on the surface a rational cubic having a node at Q. To the line corresponds a conic tangent to one of the nodal tangents of the cubic at Q. If the line determined by two points A contains B_2 and one of the 5 remaining simple points this conic becomes the corresponding line in σ and one other line. If three of the points $A_1, \cdots, A_8$ lie on a line, the surface has a new conical node. To the conic through the other five points A corresponds a rational plane quartic tangent to k at O, and having a tacnode at Q and a node at the new node of the surface. The images of the plane sections through Q and the new node are the quintics forming a pencil which have double points at the five non-collinear points A and simple points at the other three, and which pass through B_2 and hence B_2'. The images of the plane sections through O and the new node are the curves of a pencil of order 8 having triple points at the five non-collinear points A, double points at the other three, and passing through B_2 and the five simple base points. A similar situation arises if six of the points $A_1, \cdots, A_8$ lie on a conic.

THE UNIVERSITY OF CALIFORNIA

SPACES SATISFYING THE FIRST ENUMERABILITY AXIOM†

BY SELBY ROBINSON

1. *Introduction.* A neighborhood of a point p is a set of points to which p is interior, p being interior to a set V if p is a limit point of no subset of $C(V)$ and V contains p.‡ Using throughout the notation and concepts of Chittenden (loc. cit.), we define a topological space (P, K) as a collection P of points, and an undefined relation K giving for each subset E of P a unique set $K(E) = E'$ called the set of all limit points of E. Then $L(E)$ is the set of all limit points of all subsets of E; and (P, L) a space formed from (P, K) by taking as E' the set $L(E)$.

A space has property D of Hausdorff if for any pair of points there are disjoined sets to which the points are respectively interior; and is regular if, instead of for every pair of points, the property holds for every pair of disjoined sets provided each contains all its L points. Two families F and F' of neighborhoods of a point are equivalent if each set of F contains some set of F' and vice versa. For brevity we use $[U_p]$ consistently as a symbol for the family of all neighborhoods of the point p. The first enumerability axiom states that for each point p there is an enumerable family of neighborhoods equivalent to $[U_p]$. As our definition of a space V_ω, we adopt the one given by Fréchet in *Espaces Abstraits*, that is, a V space in which for each point p there is an enumerable decreasing family of neighborhoods whose product is p and which is equivalent to $[U_p]$. Fréchet had in a previous definition required in addition that a space V_ω be an L space.§ By the fourth property of Riesz,¶ we mean that

† Presented to the Society, April 3, 1931. The author is indebted to E. W. Chittenden for assistance in the preparation of this paper.

‡ Fréchet, *Esquisse d'une théorie des ensembles abstraits*, Sir Asutosh Mookerjee's Commemoration volumes, Calcutta, 1922, vol. II, p. 362; Chittenden, Transactions of this Society, vol. 31 (1929), p. 296 and p. 293. By mistake the last clause was omitted from Chittenden's definition of interiority.

§ See *Les Espaces Abstraits*, Paris, 1928, p. 216; and Transactions of this Society, vol. 19 (1918), p. 56.

¶ *Espaces Abstraits*, pp. 209–10; D. McCoy, Tôhoku Mathematical Journal,

for any set E having any two points among its limit points, there is a subset F of E having as a limit point one but not the other of the two given points. An L space is one having the first and third properties of Riesz and the property III′: if $E' > p$, there is a compact subset of E having p for a unique limit point.†

2. *Decreasing Families of Neighborhoods.* If p and q are points of a topological space, a necessary and sufficient condition that q be not in all neighborhoods of p (or in all those of an equivalent family), is that p be not in q'. A necessary and sufficient condition that for every point p the product of $[U_p]$ be p is that no single point have a limit point outside itself.

LEMMA 1. *In a V space, a necessary and sufficient condition that for each point p the product of $[U_p]$ (or of any equivalent family) be p, is the third property of Riesz.*

To prove that the third property of Riesz holds, we first notice that no point can have a limit point outside itself. In a V space p can be in E' only if in $(E-p)'$. Let E be $p+q$. Then p cannot be in E', therefore not in p'.

LEMMA 2. *A necessary condition that a point p have a decreasing family of neighborhoods equivalent to $[U_p]$ is that for any sets B and C such that $p < L(B+C)$, p is either in $L(B)$ or $L(C)$. Likewise, a necessary condition for every point to have the property is that the second property of Riesz hold in (P, L). If in the first of these theorems the point p have an enumerable family of neighborhoods equivalent to $[U_p]$, likewise if, in the second the first enumerability axiom hold, the above conditions are also sufficient.*

To prove the necessity, assume p neither in $L(B)$ or $L(C)$. Then $P-B$ and $P-C$ are neighborhoods of p and each contains a neighborhood of the decreasing family. The lesser of the two neighborhoods thus obtained is included in $P-(B+C)$. The proof of sufficiency is similar to that used in a theorem of

vol. 33 (1930), p. 90; Frink, this Bulletin, vol. 36 (1930), p. 282; Chittenden, loc. cit., p. 294. (The first E in Chittenden's statement should be E'.) See Riesz, Atti del 4 Congresso Internazionale dei Matematici, vol. 2, pp. 19–20; also Wiener, Bulletin de la Société Mathématique, vol. 50 (1922), pp. 128–129; and Fréchet, Bulletin des Sciences Mathématiques, (2), vol. 42, p. 139.

† See *Espaces Abstraits*, pp. 171–2 and 210–1; also Urysohn, L'Enseignement, vol. 25 (1926), pp. 77–82.

Fréchet.† This part of the theorem (therefore part of Theorem 1) is really due to Chittenden, who proved to me that the second is the only one of the properties of Riesz used in obtaining a decreasing family.

THEOREM 1. *If (P, K) satisfies the first enumerability axiom, a necessary and sufficient condition for every point p to have a decreasing family of neighborhoods (subfamily of the given enumerable family if desired) equivalent to $[U_p]$ and whose product is p is that no point have a limit point outside itself and the second property of Riesz hold in (P, L). Any point p is either a neighborhood of itself or has an infinity of distinct neighborhoods in any family equivalent to $[U_p]$. If a particular point has an enumerable family of neighborhoods, the theorem holds for that point.*

This follows from Lemma 2 and the first paragraph of this section. The fact that either p is a neighborhood of itself or its family F of neighborhoods is infinite, follows from the fact that the product of F is p, and if F is finite this product is a neighborhood. The sufficiency of the condition is a generalization of Fréchet's theorem cited above.

THEOREM 2. *The validity of the second property of Riesz in (P, K) implies its validity in (P, L). The analogous property holds for a single point.*

Suppose $p<L(B+C)$. Then $p<E'$ for some $E<B+C$. By our hypothesis p is either in $(E\cdot B)'$ or $(E\cdot C)'$, therefore either in $L(B)$ or $L(C)$. From this result we can get sufficient conditions to substitute for those of Lemma 2 and Theorem 1. From Lemma 2 and the equivalence of $L(E)$ and E' in a V space, we get the following lemma.

LEMMA 3. *In a V space satisfying the first enumerability axiom, a necessary and sufficient condition for the existence of the decreasing families of Lemma 2 is the second property of Riesz.*

THEOREM 3. *A necessary and sufficient condition that a space be a space V_ω is that it satisfy the first three conditions of Riesz and the first enumerability axiom.*

This follows immediately from Lemmas 1 and 3. I was led to

† American Journal of Mathematics, vol. 50 (1928), p. 62.

investigate the necessity of this condition by a remark in a letter of Orrin Frink.

3. *An Equivalence.* THEOREM 4. *In a space V_ω, the following properties are equivalent.*

I. *The fourth property of Riesz.*

II. *Property D of Hausdorff.*

III. *The property of being an L space.*

In any space, III implies III′, which implies I, so it suffices to prove that I implies II and II implies III. Suppose I holds but II is invalid. Then for some pair of points p and q, if V and W are any neighborhoods of p and q respectively, $V \cdot W \neq 0$. As no point is isolated, there is an enumerably infinite decreasing sequence V_n of distinct neighborhoods of p equivalent to $[U_p]$, and a corresponding sequence W_n for q. By omitting a finite number of neighborhoods at the beginning of each sequence, we can insure that $V_n \cdot q = 0$ and that $W_n \cdot p = 0$. Let a_n be a point of $V_n \cdot W_n$ and let $A = [a_n]$. Then p and q are points of A' but not of A. By I there is a subset B of A which has p (or q) but not q for a limit point. But as any W_n contains all but a finite number of the points of A, it contains a point of B. So $B' > q$. To prove that II implies III, we consider a set E having p for a limit point, and by the use of a decreasing sequence of neighborhoods select from E a compact subset having p for a unique limit point.

The validity of II in a space (P, K) satisfying all our hypotheses except the first property of Riesz is equivalent to the validity of I, II, or III in (P, L). If property two of Riesz holds in (P, L) (see Theorem 2), II implies I; and if the first enumerability axiom also holds, II implies III′.

Fréchet once stated that under our conditions I implies III.† Wiener published (loc. cit., pp. 131–2) an example which purported to invalidate this statement, remarking, however, that III holds if II also be assumed. Wiener's statement of the fourth postulate of Riesz, seemingly a translation of the one given by Fréchet in the article just cited, reads:

"If A is a limit-element of E, and if B is distinct from A, there is always at least one set F which has A for a limit-element with-

† Bulletin des Sciences Mathématiques, loc. cit., p. 149.

out having B for a limit-element." Wiener's example has this property if and only if F is not required to be a subset of E. The fact that II is implied by I or III seems to be new, except for the known equivalence of II and III under the additional hypothesis of the closure of derived sets.†

4. *A Correction.* Given any space V_ω containing a certain point p not isolated, we construct from this a space V_ω not an L space by adding an element p^* which is a limit point of exactly the sets having p for a limit point and which is such that for every E, $E' = (E-p)'$. Fréchet states that a space V_ω is necessarily an L space and gives another sufficient condition which is also contradicted by our example; namely, that for any set E having any point q as a limit point, there is a sequence $N<E$ such that any neighborhood of q contains all but a finite number of the points of N.‡

5. *Questions of Enumerability.* Sierpinski has stated and Putnam has generalized, certain equivalences involving the enumerability of increasing and decreasing families of closed sets.§ All the theorems of these papers extend to topological space (substituting for the separability of a set E the existence of an enumerable subset N of E such that $N+L(N)>E$, and for E being clairsemé the non-existence of $M<E$ such that for any point p in M, $L(M-p)>M$); it being necessary, however, in those theorems for which Sierpinski assumed the closure of derived sets to assume that the interior of every set is open. From this we get a theorem which we state without proof.

† Hausdorff, *Grundzüge der Mengenlehre*, first edition, Leipzig, 1914, p. 263; Fréchet, American Journal of Mathematics, loc. cit., p. 65.

‡ *Espaces Abstraits*, p. 216; Bulletin des Sciences Mathématiques, loc. cit., p. 147.

§ Sierpinski, Fundamenta Mathematicae, vol. 2 (1921), p. 179; and Putnam, this Bulletin, vol. 36 (1930), p. 653. Putnam's statement that in any V space in which derived sets are closed, Sierpinski's property II implies III and IV implies V, can be shown by an example to be incorrect. The correct statement is: V spaces in which a set plus its derived set is closed. In the proof of both these theorems, it is the closure of the sets $F_\alpha = Q_\alpha + Q_\alpha'$ which is needed, as the sets Q_α' are not known to be distinct. In Putnam's last theorem, separability should be replaced by the separability of every set. In §4, p. 183, Sierpinski does not succeed in proving the separability of every set, but merely the separability of the space. To remedy this, let the sequence (5) be, not all points of the space, but the points of some set assumed not separable.

THEOREM 5. *A space satisfies the first enumerability axiom if each point p has a well ordered decreasing family of neighborhoods equivalent to $[U_p]$, the sets $L(E)$ are closed, and either every set is separable or the space is regular and every set is condensed in itself.*

From this result we can get variations of Theorems 3 and 4.

6. *An Example.* Frink proposed to me the problem (which he and Hildebrandt had considered) of constructing a bicompact separable, completely normal, Hausdorff space, satisfying the first enumerability axiom but not the second. The circumference of a circle is such a space if we define as follows the sets which for every point p have p as a limit point. Any set E has p as a limit point if and only if there are points of $E-p$ either (1), arbitrarily near p in the direction clockwise from p, or (2), arbitrarily near q (the point opposite p) in the direction counterclockwise from q. If we omit (2), we have a locally non-compact space having the other properties mentioned.†

UNIVERSITY OF IOWA

ADDENDUM ON FACTORABILITY OF NUMERICAL FUNCTIONS

BY E. T. BELL

It has been pointed out to me that the statement of the Lemma on page 252 of the April, 1931, issue of this Bulletin might be misunderstood. To clear up any possible misunderstanding, it is sufficient to point out that the notation f, g, h is as in the italicised statement on page 251, as is evident on referring to paragraph 3, third sentence. At the top of page 253, for "*all or none of $f, g, \cdots, h$ are factorable,*" read "*the only excluded possibility is that in which precisely one of $f, g, \cdots, h$ is not factorable.*" This is evident from paragraph 3.

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† See Fréchet, Transactions of this Society, loc. cit., p. 60; and D. McCoy, loc. cit., p. 114 (S 241).

A REMARK CONCERNING THE NECESSARY CONDITION OF WEIERSTRASS*

BY E. J. MCSHANE†

Let us consider a class $\mathfrak{R}$ of rectifiable curves C lying in a point set A of n-dimensional space, and an integral $F(C)=\int_C f(x, x')ds$, where $x=(x^1, \cdots, x^n)$ and s connotes that we use the length of arc as parameter. Suppose that a certain curve $C: x=x(s)$ minimizes $F(C)$ in $\mathfrak{R}$, and denote by L the set of points of C which are interior to A and of indifference with respect to $\mathfrak{R}$ and A. Then for almost all points of L we have‡ $E(x(s), x'(s), \bar{x}')\geqq 0$ for all sets of numbers $\bar{x}'$. Given now a particular point $x(s_0)$ of L; when can we say that the inequality holds at $x(s_0)$?

It has already been shown§ that the inequality holds if $x'(s_0)$ exists, $\Sigma[x^{i'}(s_0)]^2>0$, and the $x^{i'}(s)$ are all approximately continuous at s_0. We will now show that the inequality also holds if $\Sigma(x^{i'}(s_0))^2=1$. (As is well known, this sum never exceeds 1, and is equal to 1 almost everywhere.)

Suppose then that $\Sigma[x^{i'}(s_0)]^2=1$ and that in contradiction to our statement there exists an $\bar{x}'$ such that $E(x(s_0), x'(s_0), \bar{x}')=-2k<0$. Denote by $\alpha(s)$ the angle between $x'(s)$ and $x'(s_0)$. The function

$$\phi(s) = \frac{d}{ds}\Big[\sum x^i(s)x^{i'}(s_0)\Big] = \sum x^{i'}(s)x^{i'}(s_0)$$
$$= \Big\{\sum [x^{i'}(s)]^2\Big\}^{1/2}\Big\{\sum [x^{i'}(s_0)]^2\Big\}^{1/2}\cos\alpha(s)$$

is defined for almost all values of s, and $|\phi(s)|\leqq|\cos\alpha(s)|$. By the continuity of E, we can find positive numbers ϵ, δ such that $E(x(s), x'(s), \bar{x}')<-k$ for all s such that $|s-s_0|\leqq\epsilon$, $\phi(s)\geqq 1-\delta$; and if ϵ be small enough, $x(s)$ will be in L. But $\phi(s_0)=1$ and $\phi(s)$

* Presented to the Society, April 3, 1931.

† National Research Fellow.

‡ L. Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2, p. 87. E. J. McShane, *On the necessary condition of Weierstrass*, etc., Annals of Mathematics, vol. 32.

§ E. J. McShane, loc. cit.

is a derivative; therefore* there exists on $[s_0-\epsilon, s_0+\epsilon]$ a set of positive measure for which $\phi(s)>1-\delta$, "..., which contradicts the theorem quoted above."

OHIO STATE UNIVERSITY

A CORRECTION AND AN ADDITION

BY G. E. RAYNOR

1. *A Correction.* In a former paper† by the author the minus sign on the right side of equation (4), page 888, makes the notations of equations (4) and (5) for the function G inconsistent. This difficulty may be removed by changing the sign of G throughout the paper wherever the first argument of G has r_1 in the denominator. This change makes the first footnote on page 888 superfluous and it should be deleted. The second argument of G in equations (9) and (20) should be 0 instead of θ.

2. *An Addition.* The mean value of the function Φ over the circle C_2 was considered, in the paper, for the case of the singular point P outside of C_2 and for the case of P inside of C_2. The question naturally arises as to what the situation is in case P lies on C_2. This third case is not, however, of much interest since the integral

$$\int_{C_2} \Phi ds,$$

which is now in general improper, will not in general exist. This may readily be verified for the function

$$\Phi = \left(\frac{r^2}{r_1^2} - \frac{r_1^2}{r^2}\right)\cos 2\theta$$

integrated over the circle C_2, whose equation is $\rho = r_1 \sin\theta$. It will be found that even the principal value of the above integral is infinite while of course the value of Φ at the center of C_2 is finite.

LEHIGH UNIVERSITY

* Hobson, *Theory of Functions of a Real Variable*, vol. 1, §403.

† *On the extension of the Gauss mean-value theorem to circles in the neighborhood of isolated singular points of harmonic functions*, this Bulletin, vol. 36 (1930), pp. 887–893.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

MINNEAPOLIS, MINNESOTA, SUMMER MEETING AND COLLOQUIUM, September 8-11, 1931.

NEW YORK CITY, October 31, 1931.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th Street, New York City, not later than October 10. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than September 7. By invitation of the program committee, Professor D. V. Widder will deliver an address on *Some recent developments in the theory of algebras*. Professor E. Landau will speak on *Schnirelmann's Theorem*.

COLUMBUS, OHIO, November 27-28, 1931.

Abstracts must be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 5. All abstracts received by that date will appear in the November issue of this Bulletin.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 28, 1931.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than October 31. All abstracts received by that date will appear in the November issue of this Bulletin.

NEW ORLEANS, LOUISIANA, ANNUAL MEETING, December 28-31, 1931.

Abstracts are due by November 28, 1931.

The usual February Meeting in New York City will be omitted.

NEW YORK CITY, March 25-26, 1932.

Abstracts are due by March 5, 1932.

STANFORD UNIVERSITY, CALIFORNIA, April 9, 1932.

Abstracts are due by March 12, 1932.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the

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